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A Particle Model for Spinodal Decomposition

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We study a one-dimensional lattice gas where particles jump stochastically obeying an exclusion rule and having a "small" drift toward regions of higher concentration. We prove convergence in the continuum limit to a nonlinear parabolic equation whenever the initial density profile satisfies suitable conditions which depend on the strength a of the drift. There is a critical value a_c of a. For $a < a_c$, the density values are unrestricted, while for $a \geq a_c$, they should all be to the right or to the left of a given interval $\mathscr{I}(a)$. The diffusion coefficient of the limiting equation can be continued analytically to $\mathscr{I}(a)$, and, in the interior of $\mathscr{I}(a)$, it has negative values which should correspond to particle aggregation phenomena. We also show that the dynamics can be obtained as a limit of a Kawasaki evolution associated to a Kac potential. The coefficient a plays the role of the inverse temperature β. The critical value of a coincides with the critical inverse temperature in the van der Waals limit and $\mathscr{I}(a)$ with the spinodal region. It is finally seen that in a scaling intermediate between the microscopic and the hydrodynamic, the system evolves according to an integro-differential equation. The instanton solutions of this equation, as studied by Dal Passo and De Mottoni, are then related to the phase transition region in the thermodynamic phase diagram; analogies with the Cahn–Hilliard equations are also discussed.

KEY WORDS: Hydrodynamic lattice gas; long-range interactions; phase segregation; rigorous results.

This paper is dedicated to Jerry Percus with great affection on the occasion of his 65th birthday.

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1. INTRODUCTION

There has been and continues to be much interest in simple microscopic model systems whose collective behavior is described by hydrodynamic-type equations. The reason for this is both fundamental and practical:

1. We want to have a clear understanding, hopefully a mathematical derivation, of the microscopic origins of the equations describing real macroscopic systems. For example, we want to obtain the Euler and Navier–Stokes equations for fluids from the underlying Hamiltonian microscopic dynamics of the atoms and molecules making up macroscopic matter. Our present mathematical abilities are, however, unequal to this task. We therefore consider the same problem for simplified model systems with various types of stochastic dynamics. For a discussion of the relevance of such models to real systems see ref. 17.

2. We are interested in practical methods for computing solutions of nonlinear hydrodynamic-type equations. The numerical methods developed in the past half century, while quite powerful in some cases, leave much to be desired in others. This is particularly so in the case of fluid flow at high Reynolds number and/or in complex geometries. A possible remedy to this situation is the utilization of microscopic models with the following two desiderata: (i) They have the correct macroscopic behavior, i.e., the one described by the appropriate macroscopic equations, and (ii) their microscopic dynamics can be implemented efficiently on available (or soon to be available) computers for systems large enough to exhibit macroscopic behavior. With luck this would lead to an analogue method for solving certain problems which would be better (or cheaper) than numerical methods.

The jury is still out on whether the lattice gas models introduced by Frisch *et al.*,[6] the FHP cellular automaton, and/or their descendants (see ref. 5), satisfy the above criteria in the various physical situations of interest where comparisons are appropriate and available. We shall not enter this controversy here, as our motivation is primarily fundamental. We note, however, that the model of segregating binary fluid for which we here prove hydrodynamic behavior belongs to a class in which there is much interest from both theoretical and experimental points of view; see, for instance, refs. 7, 8, 10, and 12–14.

The Model

The model we shall investigate here from a rigorous point of view is a modified (and simplified) version of a model introduced by Rothman and Keller[20] and further studied by Rothman and Zaleski[21] and Appert and

Zaleski.[1] They consider an FHP-type hexagonal lattice gas system containing two types of particles, say A and B, in which the dynamical rules are so rigged that there is a tendency for the mixture to segregate. More precisely, A and B particles on a given lattice site will have their velocities changed in a way that, consistent with momentum and species conservation, will optimize the flux of A's (B's) in the same direction as their concentration gradient. Numerical simulations clearly show segregation of the two species in regions of high total particle density and fractional concentration.[20,21] Assuming a factorization of the probabilities at different lattice sites, corresponding to local equilibrium, and making a Chapman–Enskog type of expansion of the resulting Boltzmann equation led to a nonlinear hydrodynamic equation in which there is a mutual diffusion coefficient which becomes negative in approximately the regions in which phase segregation was observed.[21] We shall make a great many simplifications in the above model, always keeping in mind, however, the central point: the macroscopic equation will have a diffusion coefficient which becomes negative in some concentration range. While some of our modifications are essential to make the model amenable to a mathematical treatment, others are mostly for convenience. The most important of the former is the introduction of a (dominant) stochastic element in the dynamics: this is essential, at the present time, to obtain the local equilibrium state in the appropriate hydrodynamic scaling limit; cf. ref. 2. A less important modification is to consider the case where the total density is maximal, so that the total momentum is identically zero for every configuration and so need not be considered explicitly. This permits us to focus attention exclusively on one of the components, say A, which we call simply particles, while the B's are the holes. The dynamics is then such that particles prefer to go uphill in the concentration gradient. We measure this gradient on a quasimacroscopic scale, i.e., to decide on its preferred direction of velocity in a given microscopic configuration, the particle looks at a large number of sites, infinite in the scaling limit, to find the direction of the concentration gradient. Finally, we limit ourselves here to a one-dimensional model.

The macroscopic equation we shall arrive at for the properly scaled microscopic density profile $f(r, t)$ will have the form

$$\frac{\partial f}{\partial t}=\frac{\partial}{\partial r}\left[D_a(f)\frac{\partial f}{\partial r}\right] \tag{1.1}$$

with $f(r, 0)=f_0(r)$, $0 \leqslant f_0(r) \leqslant 1$. Here $r \in \mathbb{R}$, $t \in \mathbb{R}^+$ are measured in macroscopic space and time units and

$$D_a(f)=\tfrac{1}{2}-af(1-f) \tag{1.2}$$

is a nonlinear diffusion constant which takes on nonpositive values for f in some interval $[I_a^-, I_a^+] \subset [0, 1]$, where $f(1-f) \geqslant (2a)^{-1}$. This can happen if the parameter $a \geqslant 2$ [for $a < 0$ the second term in (1.2) will always be positive; our analysis applies also to this case and is much simpler]. We shall in fact be able to derive (1.1) when $a \geqslant 2$ only for the case where $f_0(r)$ takes values in $[0, I_a^-)$ or $(I_a^+, 1]$, i.e., when $D_a(f) > 0$ for all values of the density between $\min f_0(r)$ and $\max f_0(r)$. This is the case when the solutions of (1.1) are at least linearly stable. In the other cases we do not have theoretical results; there are, however, recent computer simulations due to G. Giacomin on a cellular automata with the same updating rules that we consider here. They show that the system undergoes phase segregation on a space scale of the order of the interaction length, while on the macroscopic scale the density profile does not change and the effective diffusion coefficient in the unstable region vanishes.

The paper is organized as follows. In Section 2 we give a precise description of our lattice gas model. We also present there a continuous-time variation of our model for which the mathematical proofs are somewhat simpler to describe. In Section 3 we show that our evolution can be obtained from an evolution satisfying detailed balance for a Kac potential V_γ, $\gamma \approx 1/N$.[16] We can then interpret our results in terms of the thermodynamic phase diagram for V_γ, as given in the analysis of the van der Waals limit ($\gamma \to 0$). The parameter a equals the inverse temperature β, the critical value $a = 2$ is the same as the critical β for the thermodynamic free energy, while the interval $[I_-(a), I_+(a)]$ coincides with the spinodal region. The phase transition region is related to the instanton solutions of an integrodifferential equation which describes the particle system under a scaling intermediate between the microscopic and the hydrodynamic ones. In Section 4 we introduce a discrete equation, obtained from the equation for the expected value of the density at given lattice sites, assuming factorization of the probabilities. We then show that the derivation of (1.1) is reduced to proving that certain approximate solutions (which we call semi-solutions) of the above discrete equation converge to the solution of (1.1). In Section 5 we prove such a statement under the assumption that *for all the initial densities the diffusion coefficient is strictly positive.* In Sections 6 and 7 we give technical details left out from Section 5; they essentially refer to estimates on the solutions of discrete stochastic approximations of (1.1) which arise naturally when studying the microscopic structure of the evolution.

2. DESCRIPTION OF MODELS AND STATEMENT OF RESULTS

We first give a description of the lattice gas automaton model and then present a closely related continuous-time model for which the mathematical proofs are more transparent. The proofs work also for the discrete-time model, but we shall not present them here.

2.1. Discrete Time Model

We consider a system of particles on a one-dimensional lattice, each one with velocity $\sigma = \pm 1$. We denote by Ω the set of all particle configurations ξ, $\xi = \{\xi(x, \sigma): \xi(x, \sigma) \in \{0, 1\}, x \in \mathbb{Z}, \sigma(x) \in \{-1, 1\}\}$, and by $\xi(x)$ the occupation number at the site x, namely $\xi(x) = \xi(x, -1) + \xi(x, 1)$.

There is an exclusion rule that prevents two particles at the same site from having the same velocity. Therefore each site $x \in \mathbb{Z}$ can have at most two particles (with different velocities). Each unit updating of the automaton consists of two steps, the first is stochastic, the second one deterministic:

Step 1. Velocity Flips. This updating rule acts at each site x. If there are two or no particles at x, then nothing happens. If, on the other hand, there is one particle at x with velocity $\sigma(x)$, then its new velocity $\sigma'(x)$ takes the value 1 or -1 with probability p and $1-p$, respectively, where

$$p = 1/2 + \lambda_N(x; \xi)/4 \tag{2.1a}$$

and $|\lambda_N(x; \xi)| < 2$ is of the form

$$\lambda_N(x; \xi) = \sum_{i=1}^{\infty} \phi_N(i)[\xi(x+i) - \xi(x-i)] \tag{2.1b}$$

with $\phi_N(i)$ having range N (or, more generally, decaying on a space scale N, as we shall see in Section 3).

Step 2. Advection. Every particles at site x moves to $x + \sigma'(x)$ keeping its velocity.

According to (2.1), the particle at site x chooses its new velocity $\sigma'(x)$ by looking at the configuration of particles, independently of their velocities, with a weighting factor $\phi_N(i)$, in a neighborhood of size $2N$. When $\phi_N(i) > 0$ the interaction will produce a tendency toward clumping which will oppose the tendency toward uniformization produced by the diffusive term 1/2 in (2.1a). This is the effect we are after for the macroscopic behavior described by (1.1)–(1.2), i.e., $\phi_N(i) > 0$ will correspond to $a > 0$ there.

Our choice for $\phi_N(i)$ here (other possibilities will be discussed in Section 5) is

$$\phi_N(i) = \begin{cases} aN^{-2}, & i \leqslant N \\ 0, & i > N \end{cases} \tag{2.2}$$

with $N > 2\,|a|$, N will diverge in the continuum limit, as we are going to see. To obtain hydrodynamic equations we have to consider a limit in which the ratio of microscopic to macroscopic spatial scales, denoted by ε, goes to zero. To get (1.1) we look at the density profile at microscopic times of order $\varepsilon^{-2}\tau$ and set N equal to the integer part of $\varepsilon^{-1+\alpha}$, or sloppily, $N = \varepsilon^{-1+\alpha}$, with $0 < \alpha < 1$. The particle will thus sample a microscopically large but macroscopically vanishing neighborhood. We note here that we can rewrite (1.1) in the form

$$\frac{\partial f}{\partial \tau} + \frac{\partial}{\partial r}\left[f(1-f)E\right] = \frac{1}{2}\frac{\partial^2 f}{\partial^2 r}, \qquad E = a\frac{\partial f}{\partial r} \tag{2.3}$$

Equation (2.3) now resembles the Burgers equation derived in ref. 15 for the weakly asymmetric stochastic automaton process where $\lambda_n(x;\xi)$ in (2.1b) is replaced by εE independently of ξ. We obtain the Burgers equation from (2.3) when $\partial f/\partial r$ on the left-hand side of (2.3) is replaced by the externally given constant "drift velocity" there. The relationship between our (nonreversible) dynamical model and reversible models which satisfy detailed balance will be discussed in Section 3.

2.2. Continuous-Time Model

The macroscopic behavior of the discrete-time model described above is the same as that exhibited by another particle model for which proofs are somewhat simpler and more transparent, so that, in the sequel, we shall restrict our attention to this new system, which is a generalization of the weakly asymmetric simple exclusion process. The time is now continuous, the particles do not have velocities, and at each site there can be at most one particle, $\eta(x) = \{0, 1\}$. We also make a finite-volume assumption: we consider values of $\varepsilon > 0$ so that ε^{-2} is an integer and for any such ε the configuration space is

$$\Omega^{\varepsilon} = \{\eta \in \{0, 1\}^{\mathbb{Z}} : \eta(x) = \eta(x + 2\varepsilon^{-2}), \text{ for all } x \in \mathbb{Z}\} \tag{2.4}$$

In other words, the particle configurations are those of a lattice gas (with single occupancy) in a bounded interval of size $2\varepsilon^{-2} + 1$ with the identification of the first and last sites in the interval. The finite-volume assumption is not critical from a physical point of view because when $\varepsilon \to 0$ the space becomes infinite even in macroscopic units ($\equiv \varepsilon^{-1}$).

The system evolves in time by particles attempting jumps to neighboring sites. The jump rate from x to $x-1$ is

$$c(x, x-1; \eta) = \tfrac{1}{2}\eta(x)[1 - \eta(x-1)] \tag{2.5a}$$

just as in the symmetric simple exclusion process. The jump rate to the right is

$$c(x, x+1; \eta) = \eta(x)[1 - \eta(x+1)][\tfrac{1}{2} + \lambda_N(x; \eta)] \tag{2.5b}$$

where λ_N is defined by (2.1b) and (2.2) and, in this case, $|\lambda_N(x; \eta)| < 1/2$. $\lambda_N(x; \eta)$ determines an asymmetry of the jumps; if we replace it by a nonzero constant we get the asymmetric simple exclusion process; if the constant is proportional to ε, we get the so-called weakly asymmetric simple exclusion process. In our case $\lambda_N(x; \eta)$ is of order N^{-1}. If $N = \varepsilon^{-1}$, then we have a mean field interaction, which can be studied in a relatively simple way. In our case $N = \varepsilon^{-1+\alpha}$, $0 < \alpha < 1$, and the analysis becomes rather more delicate. We could equally well consider a more symmetric evolution where also the jumps to the left are modified as in (2.5b); in the present form some formulas become slightly simpler.

Summarizing, we have defined a process with state space Ω^ε, i.e., the set of configurations which are periodic with period $l \equiv 2\varepsilon^{-2}$, with generator

$$L_N f(\eta) = \sum_{x=1}^{l-1} \sum_{b=\pm 1} \sum_{n \neq 0} c(x, x+b; \eta)[f(\eta^{(x+nl, x+nl+b)}) - f(\eta)] \tag{2.6a}$$

where

$$\eta^{(x,y)}(z) = \begin{cases} \eta(z) & \text{if} \quad z \neq x, y \\ \eta(x) & \text{if} \quad z = y \\ \eta(y) & \text{if} \quad z = x \end{cases} \tag{2.6b}$$

(since the evolution is periodic, whenever a particle jumps, all the particles which are at sites differing by nl make the same jump).

2.3. The Initial Measure

The initial measure μ^ε is defined so that the variables $\eta(x)$ for $x \in [1, 2\varepsilon^{-2}]$ (ε^{-2} is an integer) are independent; it is then extended to the whole Ω^ε by imposing periodicity, namely that $\eta(x + 2\varepsilon^{-2}) = \eta(x)$. To specify the averages of the $\eta(x)$, we use the function $f_0(r)$ defined in Section 1 as the initial condition for the macroscopic equation (1.1). We assume that $f_0(r)$ is smooth, $\mathbb{C}^\infty(\mathbb{R})$, with uniformly bounded derivatives and having values in the whole interval $[0, 1]$ if $a < 2$, while if $a \geqslant 2$ its values are contained either in $[0, I_a^-)$ or $(I_a^+, 1]$; see (1.2) for notation. We then consider the function $\tilde{f}_0^\varepsilon(r)$, which is a smooth periodic approximation of $f_0(r)$. More precisely it is in $\mathbb{C}^\infty(\mathbb{R})$ with uniformly bounded derivatives and having values in the same intervals specified above. We also require that $\tilde{f}_0^\varepsilon(r) = f_0(r)$ for $|r| \leqslant \varepsilon^{-1} - 1$ and that $\tilde{f}_0^\varepsilon(r)$ is periodic with period $2\varepsilon^{-1}$. We then set

$$\mu^\varepsilon(\eta(x) = 1) = \tilde{f}_0^\varepsilon(\varepsilon x) \tag{2.7}$$

2.4. Results

We denote by $\mathbb{P}_\nu^\varepsilon$ the law of the process on Ω^ε with initial measure ν and with generator L_N. $\mathbb{E}_\nu^\varepsilon$ denotes its expectation; recall that $N = \varepsilon^{-1+\alpha}$, $0 < \alpha < 1$.

We have the following results:

Theorem 2.1. Let μ^ε, $f_0(r)$ and $\tilde{f}_0^\varepsilon(r)$, and $N = \varepsilon^{-1+\alpha}$ be as above. Then for α small enough, for any $n \geqslant 1$, for any $\tau > 0$

$$\lim_{\varepsilon \to 0} \sup_{x_1,\dots,x_n} \mathbb{E}_{\mu^\varepsilon}^\varepsilon\left(\left|\prod_{i=1}^n \eta(x_i, \varepsilon^{-2}\tau) - \prod_{i=1}^n \tilde{f}^\varepsilon(\varepsilon x_i, \tau)\right|\right) = 0 \tag{2.8}$$

where the sup is over distinct sites in $[1, 2\varepsilon^{-2}]$, $\eta(x, t)$ is the occupation number at (x, t), and $\tilde{f}^\varepsilon(r, \tau)$ is the solution of (1.1) with initial condition $\tilde{f}_0^\varepsilon(r)$. Furthermore, denoting by $f(r, \tau)$ the solution of (1.1) with initial condition $f_0(r)$, then $\tilde{f}^\varepsilon(r, \tau) \to f(r, \tau)$ uniformly on the compacts and faster than any positive power of ε.

We also have an estimate on the rate of convergence in (2.8) which allows us to estimate the structure of the single realizations of the process. For $0 < \gamma < 1$ define

$$M(x, t, \varepsilon^{-\gamma}) = \varepsilon^\gamma \sum_{|y-x| \leqslant \varepsilon^{-\gamma}/2} [\eta(y, t) - f(\varepsilon y, \varepsilon^2 t)] \tag{2.9}$$

Notice that these intervals are infinitesimal in macroscopic units $(\equiv \varepsilon^{-1})$ and that the $M(x, t, \varepsilon^{-\gamma})$ are random variables different in general from zero and different for each realization.

Theorem 2.2. We use the same notation and assumptions as in Theorem 1.1. Then for any $0 < \gamma < 1$ there is $\xi > 0$ so that for any $0 < \tau' < \tau''$ and R

$$\lim_{\varepsilon \to 0} \mathbb{P}_{\mu^\varepsilon}^\varepsilon\left(\sup_{\tau' < \varepsilon^2 t \leqslant \tau''} \sup_{|x| \leqslant \varepsilon^{-1} R} |M(x, t, \varepsilon^{-\gamma})| > \varepsilon^\xi\right) = 0 \tag{2.10}$$

and the convergence in (2.10) is faster than any power of ε.

We shall prove the above theorems in Sections 4–6; in the next section we relate our system to a reversible evolution, so that it will be possible to relate dynamical and equilibrium properties of the system.

3. REVERSIBLE MODELS, PHASE TRANSITION, AND SPINODAL DECOMPOSITION

The only cases we can treat are those where the diffusion coefficient is strictly positive for all values of the density in the initial profile; the extension to the general case requires drastically new ideas. Results in this direction would be very important since they involve fundamental questions such as the derivation of hydrodynamics in the presence of phase transitions and the introduction of particle models describing spinodal decompositions; a rigorous proof that the diffusion constant is zero in the two-phase region for some Ginzburg–Landau models with finite range has been recently obtained.[19] Given the relevance of the problem, we think we are justified if we present here some considerations which are mostly conjectures, but which might help formulate the problem in a mathematically rigorous fashion.

To discuss phase transitions it is best to use Gibbs states; to achieve this it is enough to modify a little bit the rules of our evolution. We are indebted to Herbert Spohn for helpful suggestions in this respect, as well as for many useful discussions on the whole subject presented here. The idea is that our evolution behaves as the reversible exchange dynamics associated to a Kac potential. The prototype of Kac potentials is

$$V_\gamma(r) = -\gamma e^{-\gamma r}, \qquad \gamma > 0 \quad \text{and} \quad r \geqslant 0 \tag{3.1}$$

The corresponding Hamiltonian $H_\gamma(\eta)$ is

$$H_\gamma(\eta) = \tfrac{1}{2} \sum_{x,\, y} \eta(x)\, \eta(y)\, V_\gamma(|x-y|) \tag{3.2}$$

and the Gibbs measure is

$$\mu_\gamma(\eta) = \frac{1}{Z} e^{-\beta H_\gamma(\eta)}, \qquad \beta > 0 \tag{3.3}$$

As before, the system is in a box of size ε^{-2}; the equalities (3.2) and (3.3) are therefore meaningful.

We now define a process where a particle jump obeys the exclusion rule with intensities which depend on the neighboring particles (exclusion with speed change), in such a way that the detailed balance condition with respect to μ_γ is satisfied, μ_γ being then invariant and the process reversible. This is achieved by defining the jump intensity from x to $y = x \pm 1$ as

$$c_\gamma(x, y; \eta) = \frac{1}{2} \eta(x)[1 - \eta(y)] \exp\left\{ -\frac{\beta}{2} [H_\gamma(\eta^{x,y}) - H_\gamma(\eta)] \right\} \tag{3.4}$$

where $\eta^{x,y}(z)$ is defined in (2.6b). From (3.3) and (3.4) we see that the reversibility condition

$$\mu_\gamma(\eta)\, c_\gamma(x, y; \eta) = \mu_\gamma(\eta^{x,y})\, c_\gamma(y, x; \eta^{x,y}) \tag{3.5}$$

is fulfilled. It ensures that the generator of the process with intensities given by (3.4) is self-adjoint with respect to the Gibbs measure (3.3).

By using (3.4), we have

$$\left| c_\gamma(x, y; \eta) - \frac{1}{2}\eta(x)[1-\eta(y)]\left[1 + \frac{\beta}{2}\gamma^2 \sum_z e^{-\gamma|z|} s(z)\,\eta(x+z)\right]\right| \leqslant c\gamma^2 \tag{3.6}$$

$$s(z) = \begin{cases} 1 & \text{if } z > 0 \\ -1 & \text{otherwise} \end{cases} \tag{3.7}$$

c in (3.6) is a suitable constant. Analogous formulas hold for more general potentials, like those considered by Lebowitz and Penrose,[16] and in particular when $\exp\{-\gamma|z|\}$ in (3.6) is replaced by $1\ (\gamma|z| \leqslant 1)$, i.e., the characteristic function that $\gamma|z| \leqslant 1$. In this latter case if we set $\gamma = 1/N$ we get an expression like (2.5): we therefore conjecture that our results, stated in Section 2, extend to all these cases. Notice, however, that in contrast to (2.5), also the jump intensity to the left is different from that of the symmetric exclusion process; in the continuum limit this should be equivalent to considering only modifications to jumps on the right (as in Section 2), but with $\beta/2$ in (3.6) replaced by β. Then, denoting by ρ the value of a density, we get the following condition for the diffusion coefficient to be positive:

$$\tfrac{1}{2} > \tfrac{1}{2}\rho(1-\rho)\beta \int_0^\infty dr\, e^{-r}2, \qquad \tfrac{1}{2} > \rho(1-\rho)\beta \tag{3.8}$$

as obtained heuristically from (3.6) by replacing $\eta(x+z) - \eta(x-z)$ by $2z\partial/\partial r\, \rho(r)$ $(r = \varepsilon x)$ and then letting $\gamma \to 0$. The critical value β_c is 2; for $\beta < 2$ the diffusion coefficient is positive for all values of ρ. For $\beta > 2$ there is a "forbidden interval of densities," namely

$$[I_-(\beta), I_+(\beta)] \equiv \left[\frac{1}{2} - \frac{1}{2}\left(1 - \frac{2}{\beta}\right)^{1/2}, \frac{1}{2} + \frac{1}{2}\left(1 - \frac{2}{\beta}\right)^{1/2}\right] \tag{3.9}$$

For densities outside this interval the diffusion coefficient is well defined and strictly positive. According to these considerations, we see that the parameter a in Section 2 plays the role of the inverse temperature β.

We shall now see that the interval in (3.9) also has a definite significance in the phase diagram associated to the potential (3.2). As proven in ref. 16, the infinite-volume free energy $\beta^{-1}F(\beta, \rho; \gamma)$ [for the potential in (3.2)] has a limit when $\gamma \to 0$ given by

$$F(\beta, \rho) = \mathrm{CE}\left\{F_0(\rho) - \frac{\beta}{2} b\rho^2\right\} \tag{3.10}$$

where $\mathrm{CE}\{\cdot\}$ denotes the convex envelope of $\{\cdot\}$, $F_0(\rho)$ is the free energy (times β) of the system without any potential,

$$F_0(\rho) = \rho \log \rho + (1-\rho)\log(1-\rho) \tag{3.11}$$

and

$$b = -2\int_0^\infty dr\, V_\gamma(r) \tag{3.12}$$

which equals 2 in our case.

The phase transition region is determined by the values of β and ρ for which the convex envelope in (3.10) differs from its argument. The critical value of β is

$$\sup\left\{\beta\colon \min_{0\leqslant\rho\leqslant 1} \frac{\partial^2}{\partial r^2}\left[F_0(\rho) - \beta\rho^2\right] > 0\right\} \tag{3.13a}$$

This has the same value ($\equiv 2$) as the critical parameter found before using dynamical considerations. The phase transition region, for each $\beta > 2$, is the interval

$$\left[\tfrac{1}{2} - \rho^*, \tfrac{1}{2} + \rho^*\right] \tag{3.13b}$$

where $\rho^* > 0$ solves the equation

$$\log\frac{1+2\rho^*}{1-2\rho^*} = 2\beta\rho^* \tag{3.13c}$$

This interval is strictly larger than that defined in (3.9). The interval in (3.9) is, however, relevant in the phase diagram associated to (3.10); it is in fact the set of ρ's for which the argument inside the convex envelope is concave. This is the interval of complete thermodynamic instability, usually referred to as the spinodal region. Its complement inside the interval (3.13b) is the metastable region.

The evolution therefore does not seem to distinguish the metastable region from the region where there is only a single phase; from a hydrodynamic point of view the two pure phases correspond respectively to densities smaller than $I_{-}(\beta)$ and larger than $I_{+}(\beta)$. Presumably the thermodynamically pure phases, i.e., those which have densities respectively to the right and to the left of the interval in (3.13b), have a dynamical relevance only at much longer times, when tunneling effects brought about by large deviations become important. In fact, Penrose and Lebowitz[18] have proven that the free energy associated to a restricted ensemble where the "local density values" do not exceed the metastable critical density $I_{-}(\beta)$, for instance, converges, in the limit $\gamma \to 0$, to the "metastable free energy," as given by the argument of the convex envelope in (3.10). The particle densities produced by our initial measure μ^{ε} when the density profile is completely to the right or to the left of the interval in (3.9) effectively fulfills the constraints imposed by Lebowitz and Penrose. Our results agree with their analysis since we do not see any sign of phase separation in these cases at least up to the hydrodynamic times $\varepsilon^{-2}\tau$. To see such effects we should wait for some large-density fluctuation due to interval noise, which will determine the escape from the metastable phase. This is therefore expected to occur on much longer time scales, presumably of the order of $\exp(cn)$; see ref. 18.

The question remains whether the interval (3.13b) has also some dynamical significance. The answer is positive, as we shall see, using some very recent results obtained, in a work in progress, by Dal Passo and De Mottoni. This is related to the observation that there are actually two space scales in our model, one determined by the variation length in the initial state, i.e., $\approx \varepsilon^{-1}$, the other one by the range of the interaction N (or γ^{-1}); to be specific, we shall now stick to the system of Section 2. In the hydrodynamic scaling the interaction length should become infinitesimally short and this is achieved by choosing $N = \varepsilon^{-1+\alpha}$, $\alpha > 0$. But we can also consider the other limit defined by the space-time scaling

$$x \to r = N^{-1}x; \qquad t \to \tau = N^{-2}t \tag{3.14}$$

We choose accordingly the initial measure to be a product measure with averages which vary over distances of the order of N and we prove convergence, in the same sense as in Theorem 2.1, to the equation

$$\frac{\partial}{\partial \tau}\rho(r,\tau) = \frac{\partial}{\partial r}\left\{\frac{1}{2}\frac{\partial}{\partial r}\rho(r,\tau) - a\rho(1-\rho)\int_{-1}^{1} dr'\,\rho(r+r',\tau)\,s(r')\right\} \tag{3.15}$$

where $s(r)$ denotes the sign of r.

An existence theorem holds for this equation and we have convergence for unrestricted choices of initial profile. The conjecture is that the asymptotic behavior of (3.15) does describe the hydrodynamic behavior of the system. To be more precise, define

$$\delta = \varepsilon N \tag{3.16}$$

and consider as initial condition for (3.15) a δ-dependent profile defined as

$$\rho_0^{(\delta)}(r) = \bar{\rho}(\delta r) \tag{3.17}$$

where $\bar{\rho}$ is some given smooth function with values in [0, 1]. Call then $\rho^{(\delta)}(r, t)$ the corresponding solution of (3.15) and define

$$\rho_\delta(r, t) = \rho^{(\delta)}(\delta^{-1} r, \delta^{-2} t) \tag{3.18}$$

We expect for δ small that ρ_δ describes the true behavior of the particle system in the hydrodynamic regime. Since ρ_δ satisfies the equation

$$\frac{\partial}{\partial t} \rho_\delta(r, t) = \frac{\partial}{\partial r} \left\{ \frac{1}{2} \frac{\partial}{\partial r} \rho_\delta(r, t) - a \rho_\delta (1 - \rho_\delta) \, \delta^{-2} \int_{-\delta}^{\delta} dr' \, \rho_\delta(r + r', t) \, s(r') \right\} \tag{3.19}$$

with initial condition $\bar{\rho}$; the problem is then to find the limiting behavior at fixed finite times of the solution to (3.19) in the limit when $\delta \to 0$. If $\bar{\rho}$ does not have values in the interval (3.9), as assumed in Theorem 2.1, then (3.19) converges to (1.1). When, on the other hand, $\bar{\rho}$ has values in the interval (3.9), then (3.19) should describe the separation of phases. In this frame Dal Passo and De Mottoni have studied the question of the shape of the profile which connects the two phases after their separation, namely those stationary solutions to (3.19) which have different asymptotics at $\pm\infty$. They look for solutions such that, for $r \geqslant 0$, $\rho_\delta(r) \geqslant 1/2$, and for $r \leqslant 0$

$$\rho_\delta(r) - \tfrac{1}{2} = \tfrac{1}{2} - \rho_\delta(-r) \tag{3.20}$$

They prove that besides $\rho_\delta = 1/2$ there is another solution if and only if there is $u^* > 0$ such that

$$\log \frac{1 + u^*}{1 - u^*} = a u^* \tag{3.21}$$

and in such a case

$$\lim_{r \to \pm\infty} \rho_\delta(r) = \frac{1}{2} \pm \frac{u^*}{2} \tag{3.22}$$

Notice that u^* depends on a (i.e., on the temperature), but it does not depend on δ (actually it is independent of the Kac potential we are considering, provided it satisfies some suitable decay conditions). The asymptotic values of the density are therefore the endpoints of the interval (3.13b), namely the stationary solution ρ_δ connects the densities of the thermodynamically pure phases at the phase transition. Furthermore, as a consequence of the scaling properties of ρ_δ [cf. (3.19)] it follows that for $\delta \to 0$ ρ_δ converges to the step function $1/2 + u^* s(r)/2$, as shown by Dal Passo and De Mottoni.

The condition (3.21) arises also in the Cahn–Hilliard equation

$$\frac{\partial}{\partial t}\psi(r, t) = \frac{\partial^2}{\partial r^2}\left\{-c\frac{\partial^2}{\partial r^2}\psi(r, t) + f'(\psi(r, t))\right\} \tag{3.23}$$

where c is some positive constant, f' is the derivative of f, and f is a function corresponding to some local mean-field free energy density, hence not necessarily convex; see, for instance, ref. 14 for a phenomenological derivation of the equation and ref. 22 for a more mathematical discussion. Notice first that the equation does not change if f is modified by a term which is linear. To relate somehow (3.23) to our model we then choose

$$f(\psi) = F_0(\psi) - \tfrac{1}{2}\beta b(1/2 - \psi)^2 \tag{3.24}$$

which is the same as in (3.13) except for the linear terms. In this way it becomes symmetric with respect to 1/2, just the same symmetry existing in the particle evolution. Finally recall that in our case $b = 2$.

We can now look for stationary solutions which satisfy the condition

$$-c\frac{\partial^2}{\partial r^2}\psi(r, t) + f'(\psi(r, t)) = 0 \tag{3.25}$$

and are antisymmetric without being identically equal to 0. The condition for this to occur is that f has a concave part and this gives the same condition found before for the critical temperature. For temperatures below the critical one ($\beta > 2$) the asymptotic values of the stationary solution are given by the minima of the free energy; hence they define the same interval (3.13b). Of course the specific form of the solution is different from that arising from (3.19); however, under the space scaling (3.14) they converge to the same step function found before. This suggests that under the space-time scalings (3.14) the integrodifferential equation (3.19) and the Cahn–Hilliard equations might have similar behaviors.

While the above discussion establishes some relation between our model, the phase diagram, and spinodal decomposition, an analysis of what happens dynamically in the phase transition region is still lacking. Something can be said at a general level; we note first that the Green–

Kubo formula for the diffusion coefficient predicts that D is proportional to the inverse of the compressibility,[22] so that one would expect that in the phase transition region the diffusion coefficient vanishes. Such a conclusion is confirmed in ref. 19, where a Ginzburg–Landau evolution for a system with short-range interactions is studied. This is also supported by numerical experiments; see, for instance, ref. 10. In our case, considering γ as fixed, the Green–Kubo diffusion coefficient is

$$D_{\mathrm{GK}}(\gamma) = \chi_\gamma^{-1} \Bigg[\mathbb{E}_{\mu_\gamma}(j_\gamma(0, 1; \eta_0)[\eta_0(0) - \eta_0(1)]) - 2\int_0^\infty dt \sum_x \mathbb{E}_{\mu_\gamma}(j_\gamma(0, 1; \eta_0)\, j_\gamma(x, x+1; \eta_t)) \Bigg] \tag{3.26}$$

where η_t denotes the configuration at time t, χ_γ is the compressibility coefficient

$$\chi_\gamma = \mathbb{E}_{\mu_\gamma}\left(\eta_0(0) \sum_x [\eta_0(x) - \mathbb{E}_{\mu_\gamma}(\eta_0(x))] \right) \tag{3.27a}$$

while

$$j_\gamma(x, x+1; \eta) = c_\gamma(x, x+1; \eta) - c_\gamma(x+1, x; \eta) \tag{3.27b}$$

is the expected "current" through the bond $x, x+1$.

A proof that the process behaves diffusively in the hydrodynamic limit, $\varepsilon \to 0$, and that the corresponding diffusion coefficient is given by the Green–Kubo formula is, however, missing. The main problem is that in general the system is "nongradient," namely the current cannot be expressed as the lattice derivative of a continuous function: the system is gradient if

$$j_\gamma(x, x+1; \eta) = h_\gamma(x+1; \eta) - h_\gamma(x; \eta) \tag{3.28}$$

with $h_\gamma(x; \eta)$ being the shift by x of a continuous function, $h_\gamma(0; \eta)$. Notice that if the system is gradient, the integral in (3.26) vanishes, so that the diffusion coefficient is given by the equilibrium average of a continuous function. For gradient systems one can use the general approach of Guo *et al.*[9] to derive hydrodynamics. In the nongradient case new problems arise which have not been solved, so that a derivation of hydrodynamics is lacking.

While it is not clear how the Green–Kubo diffusion coefficient behaves in the limit $\gamma \to 0$, some general conclusions can be drawn. The integral term in the definition of $D_{\mathrm{GK}}(\gamma)$ is positive and, for each $\gamma > 0$, it is not larger than the first term on the right-hand side of (3.26), because D_{GK} cannot be negative. Since this remains bounded when $\gamma \to 0$ while the compressibility coefficient $\chi(\gamma) \to \infty$ as $\gamma \to 0$ for values of ρ and β in the phase transition region, then for such values of ρ and β, $D_{GK}(\gamma) \to 0$. This contrasts with our results, because our diffusion coefficient is nonzero outside the spinodal region (i.e., up to and including the metastable region), but indeed there was an interchange of limits; Green–Kubo corresponds to taking first $t \to \infty$ and then $\gamma \to 0$, while in our process $\gamma = \varepsilon^{1-\alpha}$ and time diverges like ε^{-2}.

4. OUTLINE OF PROOFS

We consider the model introduced in Section 2. The proofs of Theorems 2.1 and 2.2 are essentially based on a perturbative analysis of the semigroup generated by L_N. In the traditional approach, which goes back to Boltzmann and to the derivation of the Boltzmann equation, one tries first to prove propagation of chaos. Remember that for large N, L_N looks like a "small" perturbation of L_0, the generator of the symmetric simple exclusion process, for which the Bernoulli measures are the only extremal invariant measures. If propagation of chaos holds, we can establish the relation between the process and the limiting macroscopic equation. By using propagation of chaos we can in fact estimate the right-hand side in the identity

$$\frac{d}{dt}\mathbb{E}^{\varepsilon}(\eta(x,t)) = \mathbb{E}^{\varepsilon}(L_N \eta(x,t)) \tag{4.1}$$

In this way we prove that $\mathbb{E}^{\varepsilon}(\eta(x,t))$ is close to $\rho(x,t)$, where

$$\frac{\partial}{\partial t}\rho(x,t) = -[J(x,x+1;t) - J(x-1,x;t)] \tag{4.2a}$$

and the average current $J(x, x+1; t)$ through the bond $x, x+1$ equals

$$J(x,x+1;t) = -\tfrac{1}{2}(\nabla_1^+ \rho)(x,t) + a\rho(x,t)[1-\rho(x+1,t)]\, A(x,t) \tag{4.2b}$$

where

$$A(x,t) = \frac{1}{N^2}\sum_{i=1}^{N} [\rho(x+i,t) - \rho(x-i,t)] \tag{4.2c}$$

Here and in the sequel we denote by $\nabla_j^{\pm}$, $j \geq 1$, the j-discrete derivative, i.e.,

$$\nabla_j^{\pm} f(x) = \pm j^{-1}[f(x \pm j) - f(x)] \tag{4.2d}$$

We also set Δ_j as the j-discrete Laplacian, i.e.,

$$\Delta_j f(x) = \nabla_j^+ \nabla_j^- f(x) = \nabla_j^- \nabla_j^+ f(x) \tag{4.2e}$$

It only remains at this point to prove that the solution of (4.2) converges to the solution of (1.1) when $\varepsilon \to 0$.

In several cases it has been possible to pursue this approach to the end, deriving in this way the macroscopic equations. In other cases, as in the analysis of several deterministic cellular automata, HHP and FHP, for instance, the mathematics looks too hard for a complete proof, yet the approach gives at least the possibility to "guess the right" hydrodynamic equations and to relate their transport coefficients to the microscopic features of the model.

Unfortunately, in our case the whole argument as outlined so far leads to serious technical difficulties, just as in ref. 15, where a cellular automaton simulating the Burgers equation was studied. We follow the approach used in ref. 15 and also in the analysis of several other models (see ref. 3), which essentially reverses the previous strategy, proving first that the average occupation numbers are "close" to the solution of (4.2), and only then that propagation of chaos holds. In doing this we need to and shall characterize some support properties of the process, proving that not only the average occupation numbers, but the configurations themselves are "close" to the solution of (4.2).

Let us be more precise. For any function f on Ω^ε we set

$$\| f \| = \sup_x \left| \sum_y P_{\varepsilon^{-1/2}}(x \to y)\, f(y) \right| \tag{4.3}$$

where P_t is the semigroup associated to a single simple symmetric random walk which jumps on its nearest neighbor sites with intensity 1. Therefore $\| f \|$ is approximately the sup norm of $\bar{f}$, $\bar{f}$ being obtained by averaging f over intervals of size $\varepsilon^{-1/4}$, intervals which are infinitesimally small with respect to the macroscopic scale ε^{-1}. We then introduce a time grid T, which depends on ε: $T = \varepsilon^{-2+\beta}$ and $\beta > 0$ is chosen larger than 2α, so that $N^{-1}\sqrt{T} \to 0$. Because $\lambda_N \approx N^{-1}$ we have a mean field interaction and in such a case the process can be investigated in a very accurate way; see Proposition 4.3 below. Since we shall look at the process at times which are integer multiples of T, it is convenient to redefine T as

$$T = u\varepsilon^{-2+\beta}, \qquad u \in [1, 2] \tag{4.4}$$

In this way we can represent any $t \geqslant \varepsilon^{-2+\beta}$ as an integer multiple of T, for some value of u.

Given T as above, we define a new process with values in Ω^ε. Given a realization of the original process, we denote by

$$\underline{\eta} = \{\eta_0, \eta_T, \eta_{2T}, \dots\} \tag{4.5}$$

the sequence of configurations η_{nT} at times nT. If the process were replaced by the deterministic process defined via (4.2), then each $\eta_{(n+1)T}$ would be the solution at time T of (4.2) with initial datum η_{nT}. With this in mind, we introduce the following definition. For any $\underline{\eta}$ as in (4.5) (and distributed with the law of the process having generator L_N) we define $\rho(x, t\,|\,\underline{\eta})$ so that at the various times nT is just coincides with η_{nT}, while in the time intervals $[nT, (n+1)T)$ it solves (4.2) with initial condition η_{nT}. Then $\rho(\cdot, t\,|\,\underline{\eta})$ is a new stochastic process whose discontinuities measure the difference between our original process and the deterministic one defined by (4.2). We shall say that $\rho(x, t\,|\,\underline{\eta})$ is "a semisolution of (4.2)" in the time interval $[0, S]$, if it is ε^ζ-quasicontinuous in $[0, S]$, i.e., if its right and left limits differ at most by ε^ζ: namely, $\rho(x, t\,|\,\underline{\eta})$ is ε^ζ-quasicontinuous in the time interval $[0, S]$ if for each n such that $nT \leqslant S$

$$\lim_{t \nearrow nT} \|\rho(\cdot, t\,|\,\underline{\eta}) - \eta_{nT}\| \leqslant \varepsilon^\zeta \tag{4.6}$$

Our main result is that the trajectories of the ρ-process are semisolutions of (4.2) with probability which goes to 1 as $\varepsilon \to 0$. More precisely, we have the following result.

Proposition 4.1. There exists ζ positive so that for any $2\alpha < \beta$, $u \in [1, 2]$, and $\tau > 0$ the probability that ρ is not ε^ζ-quasicontinuous in the time interval $[0, \varepsilon^{-2}\tau]$ vanishes faster than any power of ε.

For the proof we refer to refs. 3 and 4, where similar properties are proven. Actually Proposition 4.1 is a corollary of an estimate on the v-functions stated in Proposition 4.3 below.

We have now the purely analytical problem of showing that all the semisolutions of (4.2) converge to the solution of (1.1) when $\varepsilon \to 0$. We have indeed avoided the *a priori* proof of propagation of chaos, but we have paid a price, namely the problem of proving that the semisolutions and not only the solutions of (4.2) converge as $\varepsilon \to 0$ to the solution of (1.1).

By the assumptions on the initial measure μ^ε and by using the Chebitchev inequality, it is not difficult to see that for any k there is c_k so that

$$\mu^\varepsilon(\|\bar{J}_0^\varepsilon(\varepsilon x) - \eta_0(x)\| > \varepsilon^\zeta) < c_k \varepsilon^k \tag{4.7}$$

[cf. (2.7)]. We can therefore neglect the cases where η_0 in $\underline{\eta}$ is not ε^ζ close to the initial density profile, in the sense of (4.7). We shall prove in the next section the following crucial proposition.

Proposition 4.2. There exists $\gamma>0$ such that for any $\tau>0$ and for all $n\geqslant 1$ such that $nT\leqslant\varepsilon^{-2}\tau$

$$\lim_{t\nearrow nT}\ \sup_x |\mathcal{J}^\varepsilon(\varepsilon x, \varepsilon^2 t)-\rho(x, t\,|\,\underline{\eta})|\leqslant c\varepsilon^\gamma \tag{4.8}$$

for any semisolution $\rho(x, t\,|\,\underline{\eta})$ of (4.2) such that $\|\mathcal{J}_0^\varepsilon(\varepsilon x)-\eta_0(x)\|\leqslant\varepsilon^\zeta$. [The constant c in (4.8) may depend on τ.]

To derive propagation of chaos and to conclude the proof of Theorems 2.1 and 2.2, we still need another ingredient, the estimate on the v-functions which is given below in Proposition 4.4. The v-functions are defined as follows. Let $\underline{x}$ be a subset of $[1, 2\varepsilon^{-2}]$ consisting of $n\geqslant 1$ distinct sites, $x_1,..., x_n$. Then

$$v_n^\varepsilon(\underline{x}, t\,|\,\eta)=\mathbb{E}_\eta^\varepsilon\left(\prod_{i=1}^n\,[\eta(x_i, t)-\rho(x_i, t\,|\,\eta)]\right) \tag{4.9}$$

where $\rho(x, t\,|\,\eta)$ denotes the solution of (4.2) with initial datum η, $\mathbb{E}_\eta^\varepsilon$ the expectation when the initial configuration is η, and $\mathbb{P}_\eta^\varepsilon$ its law. An exponential decay on n for the v-functions has been proven for the symmetric simple exclusion process (with ρ defined accordingly) and for other processes which are "small perturbations" of the exclusion process. The proofs extend easily to our case, so we state without proof the following proposition.

Proposition 4.3. For any $\beta>2\alpha$ the following holds. For any n there exists c_n so that

$$|v_n^\varepsilon(\underline{x}, t\,|\,\eta)|\leqslant c_n t^{-n/8} \tag{4.10}$$

for all $\underline{x}$ consisting of n distinct sites, for all η and for all $t\leqslant\varepsilon^{-2+\beta}$.

Proposition 4.1 is a corollary of Proposition 4.3. One needs in fact to compute

$$\mathbb{P}_\eta^\varepsilon\left(\left|\sum_y P_{\varepsilon^{-1/4}}(x\to y)[\eta(y)-\rho(y, T\,|\,\eta)]\right|>\varepsilon^\zeta\right) \tag{4.11}$$

By using the Chebitchev inequality with arbitrary high moment and the estimate (4.10) we prove that if $\zeta<1/8$ the probability in (4.11) is smaller that any positive power of ε. We can then extend the result to arbitrary x;

here we use that the volume size is finite, $2\varepsilon^{-2}$. In the same way we obtain the uniformity over times $\leqslant \varepsilon^{-2}\tau$.

Proof of Theorem 2.1. Proof of (2.8). Given $\tau > 0$, we fix u [cf. (4.4)] so that $\varepsilon^{-2}\tau = (K+1)T$. We take the conditional expectation by fixing

$$\eta_{(lT)_{l \leqslant K}}$$

We then have

$$\mathbb{E}^{\varepsilon}_{\mu^{\varepsilon}}\left(\prod_{i=1}^{n} \eta(x_i, (K+1)T)\right)$$
$$= \mathbb{E}^{\varepsilon}_{\mu^{\varepsilon}}\left(\mathbb{E}^{\varepsilon}_{\mu^{\varepsilon}}\left(\prod_{i=1}^{n} \eta(x_i, (N+1)T) \,|\, \eta_{(lT)_{l \leqslant K}}\right)\right) \tag{4.12}$$

We add and subtract $\rho(x_i, (K+1)T) \,|\, \underline{\eta})$, where

$$\underline{\eta} = \{\eta_0, \dots, \eta_{KT}\}$$

By (4.10) we have

$$\lim_{\varepsilon \to 0} \sup_{x_1, \dots, x_n} \left| \mathbb{E}^{\varepsilon}_{\mu^{\varepsilon}}\left(\prod_{i=1}^{n} \eta(x_i, (K+1)T)\right) - \mathbb{E}^{\varepsilon}_{\mu^{\varepsilon}}\left(\prod_{i=1}^{n} \rho(x_i, (K+1)T \,|\, \underline{\eta})\right)\right| = 0$$

By using Propositions 4.1 and 4.2 we obtain the proof of (2.8).

We prove that for any k, R, and T there is a c such that for all ε small enough

$$\sup_{|r| \leqslant R} \sup_{t \leqslant T} L^k \,|\tilde{f}^{\varepsilon}(r, t) - f(r, t)| \leqslant c \tag{4.13a}$$

where

$$L := \varepsilon^{-1} - 1 - R \tag{4.13b}$$

We refer to the literature (see, for instance, ref. 11) for an existence and uniqueness theorem for (1.1) (under the assumptions we have on the initial data) and for a proof that such a solution is a $\mathbb{C}^{\infty}$ function bounded with all its derivatives. We believe that also (4.13a) can be found in the literature, but we cannot give precise references. In any case, proving (4.13a) is a useful warmup for the more intricate proof of Proposition 4.2.

We start by an integral representation for the solution to (1.1), known as the parametrix representation[11] (space and time below should always be considered as macroscopic):

$$f(x, t) = \int_{\mathbb{R}} dy\, \mathscr{G}^{(x,t)}(y-x, t)\, f_0(y) \tag{4.14a}$$
$$+ \int_0^t ds \int_{\mathbb{R}} dy\, \mathscr{G}^{(x,t)}(y-x, t-s) \frac{\partial}{\partial y} \left\{ [D(y, s) - D(x, t)] \frac{\partial f(y, s)}{\partial y} \right\}$$

where

$$\mathscr{G}^{(x,t)}(z, s) = \frac{1}{[2\pi D(x, t) s]^{1/2}}\, e^{-z^2/2D(x,t)s} \tag{4.14b}$$

$$D(x, t) = \frac{1}{2} - af(x, t)[1 - f(x, t)] \tag{4.14c}$$

We can write an analogous equation for $\bar{f}^\varepsilon$, namely

$$\bar{f}^\varepsilon(x, t) = \int_{\mathbb{R}} dy\, \mathscr{G}^{(x,t)}(y-x, t)\, \bar{f}_0^\varepsilon(y) \tag{4.15}$$
$$+ \int_0^t ds \int_{\mathbb{R}} dy\, \mathscr{G}^{(x,t)}(y-x, t-s) \frac{\partial}{\partial y} \left\{ [\tilde{D}(y, s) - D(x, t)] \frac{\partial \bar{f}^\varepsilon(y, s)}{\partial y} \right\}$$

where $\tilde{D}$ is defined as in (4.14c) with f replaced by $\bar{f}^\varepsilon$. We rewrite (4.15) by adding and subtracting $D(y, s)$ in the difference $\tilde{D}(y, s) - D(x, t)$. The time integral term in (4.15) becomes then, after integrating by parts,

$$\int_0^t ds \int_{\mathbb{R}} dy \left(\bar{f}^\varepsilon(y, s) \frac{\partial}{\partial y} \left\{ [D(y, s) - D(x, t)] \frac{\partial}{\partial y} \mathscr{G}^{(x,t)}(y-x, t-s) \right\} \right.$$
$$\left. - [\tilde{D}(y, s) - D(y, s)] \frac{\partial \bar{f}^\varepsilon(y, s)}{\partial y} \frac{\partial}{\partial y} G^{(x,t)}(y-x, t-s) \right)$$

An analogous expression is obtained from (4.14a). We write

$$g^\varepsilon(x, t) = |\bar{f}^\varepsilon(x, t) - f^\varepsilon(x, t)| \tag{4.16}$$

and after taking the difference between the expressions for f and $\bar{f}^\varepsilon$, we obtain

$$g^\varepsilon(x, t) \leqslant \int_{\mathbb{R}} dy\, \mathscr{G}^{(x,t)}(y-x, t)\, \tilde{g}^\varepsilon(y, 0)$$
$$+ \int_0^t ds \int_{\mathbb{R}} dy\, \mathscr{N}^{(x,t)}(y-x, t-s)\, g^\varepsilon(y, s) \tag{4.17}$$

where

$$\mathcal{N}^{(x,t)}(y-x, t-s) = \frac{c}{(t-s)^{1/2}} \sum_{i=1}^{3} \left[\frac{|y-x|}{(t-s)^{1/2}}\right]^i \mathcal{G}^{(x,t)}(y-x, t-s) \tag{4.18}$$

and c is a constant whose value changes from line to line. To derive (4.17) and (4.18) we have used the following estimates:

$$|D(y,s) - D(x,t)| \leqslant c[t-s+|y-x|]$$

[in fact, by the maximum principle we know that the solutions of (1.1) are bounded from above and below by the same bounds they have at time zero: therefore, from the assumptions on the initial conditions we know that $D(x,t)$ is bounded from below away from zero, while it is obviously $\leqslant 1/2$]

$$\frac{\partial^2}{\partial y^2} \mathcal{G}^{(x,t)}(y-x, t-s) = \frac{1}{D(x,t)(t-s)} \left[\frac{|y-x|^2}{D(x,t)(t-s)} - 1\right] \mathcal{G}^{(x,t)}(y-x, t-s)$$

$$\left|[D(y,s) - D(x,t)] \frac{\partial^2}{\partial y^2} \mathcal{G}^{(x,t)}(y-x, t-s)\right| \leqslant c \frac{1}{(t-s)^{1/2}} \left[\sum_{i=1}^{3} \left(\frac{|y-x|}{(t-s)^{1/2}}\right)^i\right] \mathcal{G}^{(x,t)}(y-x, t-s)$$

(recall that $t \leqslant T$)

$$\left|\left\{\frac{\partial}{\partial y}[D(y,s) - D(x,t)]\right\} \frac{\partial}{\partial y} \mathcal{G}^{(x,t)}(y-x, t-s)\right| \leqslant c \frac{1}{(t-s)^{1/2}} \frac{|y-x|}{(t-s)^{1/2}} \mathcal{G}^{(x,t)}(y-x, t-s)$$

$$\left|[\tilde{D}(y,s) - D(y,s)] \frac{\partial \tilde{f}^{\varepsilon}(y,s)}{\partial y} \frac{\partial}{\partial y} G^{(x,t)}(y-x, t-s)\right| \leqslant c \frac{g^{\varepsilon}(y,s)}{(t-s)^{1/2}} \frac{|y-x|}{(t-s)^{1/2}} \mathcal{G}^{(x,t)}(y-x, t-s)$$

We iterate (4.17), obtaining a series of terms, the nth one being

$$\int_0^t ds_1 \cdots \int_0^{s_{n-1}} ds_n \int_{\mathbb{R}} dy_1 \cdots \int_{\mathbb{R}} dy_{n+1} \times \left[\prod_{j=1}^{n} \mathcal{N}^{(y_{j-1}, s_{j-1})}(y_j - y_{j-1}, s_j - s_{j-1})\right] \times \mathcal{G}^{(y_n, s_n)}(y_{n+1} - y_n, s_n)\, g^{\varepsilon}(y_{n+1})$$

where $y_0 = x$, $s_0 = t$. We now remark that $g^\varepsilon(y_{n+1}, 0) = 0$ unless $|y_{n+1}| > L + R$ [cf. (4.13)]. Then the above term will vanish if all

$$|y_{j+1} - y_j| < \frac{L}{n+1} \tag{4.19}$$

so that we can bound it by replacing $g^\varepsilon(y_{n+1}, 0)$ by a suitable constant times the characteristic function that for at least one value of j the condition in (4.19) is not fulfilled. Therefore the nth term above is bounded by

$$c^n(n+1) \exp\left\{-\left(\frac{L}{n+1}\right)^2 \frac{1}{4Dt}\right\}$$
$$\times \int_0^t ds_1 \cdots \int_0^{s_{n-1}} ds_n\, (t-s_1)^{-1/2} \cdots (s_{n-1} - s_n)^{-1/2} \tag{4.20}$$

where D is the infimum of $D(x, t)$. We multiply (4.20) by L^k [cf. (4.13)] and we get the following upper bound for the resulting expression:

$$c^n(n+1)[(n+1)^2\, 4DT]^k\, C$$
$$\times \int_0^t ds_1 \cdots \int_0^{s_{n-1}} ds_n\, (t-s_1)^{-1/2} \cdots (s_{n-1} - s_n)^{-1/2} \tag{4.21a}$$

where

$$C = \max_{x \geqslant 0} x^k e^{-x^2} \tag{4.21b}$$

Since the n-fold integral in (4.21a) equals

$$t^{n/2}\pi^m \begin{cases} (m!)^{-1} & \text{if } \ n = 2m \\ 2^{m+1}[(2m+1)!!]^{-1} & \text{if } \ n = 2m+1 \end{cases} \tag{4.21c}$$

then the sum over n of (4.21) is finite and bounded for $t \leqslant T$. We have thus completed the proof of (4.13) and of Theorem 2.1. ∎

Proof of Theorem 2.2. The proof is very similar to that in ref. 15 for an analogous property, so we shall just outline it. The first step is to control the sup. Since time is continuous, we introduce a time grid of length θ; θ will be chosen very small, i.e., ε to some high power. Let $\mathscr{A}$ be the following event: there is a time interval $I = [n\theta, (n+1)\theta]$ which has a nonempty intersection with the time interval $[\varepsilon^{-2}\tau', \varepsilon^{-2}\tau'']$ and in I it happens twice that a particle moves starting from a site in the interval centered at the origin and of size $4\varepsilon^{-1}R$. Since the jump intensities are bounded; we get that

$$\mathbb{P}^\varepsilon(\mathscr{A}) \leqslant c(\theta\varepsilon^{-1})^2 \frac{\varepsilon^{-2}}{\theta}$$

where the constant c depends on R and τ''. By taking θ equal to ε to a sufficiently high power we can make the probability of $\mathscr{A}$ smaller than any given power of ε. Using this, we can replace (2.10) by

$$\lim_{\varepsilon \to 0} \mathbb{P}^{\varepsilon}_{\mu^{\varepsilon}}\Big(\sup_{t = n\theta:\, \tau' \leqslant \varepsilon^2 t \leqslant \tau''} \ \sup_{|x| \leqslant \varepsilon^{-1} R} |M(x, t, \varepsilon^{-\gamma})| > 2\varepsilon^{\xi}\Big) = 0 \tag{4.22}$$

because outside $\mathscr{A}$ in a time interval $[n\theta, (n+1)\theta]$ the M's in (4.22) can change at most by the displacement of just one particle by one site. By choosing $\gamma/2 > \xi$ we then obtain (4.22), at least for ε small enough.

By taking the sup out of the probability, we are reduced to proving that

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^{-2}\tau''}{\theta} 2\varepsilon^{-1} R \sup_{t = n\theta:\, \tau' \leqslant \varepsilon^2 t \leqslant \tau''} \ \sup_{|x| \leqslant \varepsilon^{-1} R} \mathbb{P}^{\varepsilon}_{\mu^{\varepsilon}}(|M(x, t, \varepsilon^{-\gamma})| > 2\varepsilon^{\xi}) = 0 \tag{4.23}$$

We estimate the above probability by using the Chebitchev inequality with power $2n$. Since M is a sum of terms, we will obtain a sum over $2n$ sites, $y_1, \ldots, y_{2n}$, of products of $2n$ factors of the form $[\eta(y_i, t) - f(\varepsilon y_i, \varepsilon^2 t)]$; cf. (2.9). There are two extreme cases; one is when the y_i are pairwise equal, in which case we obtain a bound of the form $c\varepsilon^{-2n\xi + n\gamma}$: if $\gamma/2 > \xi$, by taking n large enough we can make this term win against the diverging factors in (4.23).

The other extreme case is when all the y_i are different (all the other cases can be examined combining the arguments needed for the two extreme ones). Given t, we choose $u \in [1, 2]$ so that $t = (K+1)T$ and $T = u\varepsilon^{-2+\beta}$. By using Proposition 4.1 we can neglect the $\underline{\eta}$ for which $\rho(x, t \mid \underline{\eta})$ is not a semisolution. For the others we can use Proposition 4.2 [assuming that γ in (4.8) is larger than ξ] and the first part of Theorem 2.1 (already proven) to replace $f(\varepsilon x, \varepsilon^2 t)$ by $\rho(x, t \mid \underline{\eta})$, where the sequence $\underline{\eta}$ is stopped at KT. We now take the conditional expectation on the process up to time KT and we use Proposition 4.4. We need to assume now that $\xi < 1/8$. In this way, by choosing n, i.e., the power in the Chebitchev inequality, large enough we obtain the desired estimate. We leave out the details. ∎

5. PROOF OF PROPOSITION 4.2

To prove Proposition 4.2 we use the parametrix method, proceeding as in the proof of Theorem 2.1. We are indebted to S. Molchanov for suggesting this approach and for many helpful discussions. We do not have here the continuous equation (1.1), but a discrete approximation, Eq. (4.2):

nevertheless the method extends easily to this case. We shall write an integral equation for the solution to (4.2) in terms of a semigroup generated by replacing the nonlinear term $\rho(1-\rho)$ by $f^\varepsilon(1-f^\varepsilon)$, where f^ε is related to the solution of (1.1) and it is computed at a conveniently fixed space-time point, namely we are "freezing the environment." This is not the whole game, because we need to study not only the solutions, but also the semisolutions of (4.2); cf. Section 4 for notation. This, however, will only give an extra term in the integral equation that we can control by choosing β, hence α, small enough.

We start by showing that the solution to (1.1) solves approximately an equation similar to (4.2).

Lemma 5.1. Let $\tilde{f}^\varepsilon(r, \tau)$ be as in Theorem 2.1. We set

$$f^\varepsilon(x, t) = \tilde{f}^\varepsilon(\varepsilon x, \varepsilon^2 t)$$

and we define

$$\begin{aligned}
\Phi^\varepsilon(x, t) = {} & \sum_z G_N^{(x,t)}(z-x, t)\, f^\varepsilon(z, 0) \\
& + a \int_0^t ds \sum_z G_N^{(x,t)}(z-x, t-s)[F^\varepsilon(x, t) - F^\varepsilon(z, s)]\, \Delta_N f^\varepsilon(z, s) \\
& - a \int_0^t ds \sum_z G_N^{(x,t)}(z-x, t-s) \\
& \times \frac{1}{N^2} \sum_{i=1}^N [f^\varepsilon(z-1+i, s) - f^\varepsilon(z-1-i, s)]\, \nabla_1^- F^\varepsilon(z, s)
\end{aligned} \tag{5.1}$$

where ∇_1^- and Δ_N are defined in (4.2),

$$F^\varepsilon(z, s) = f^\varepsilon(z, s)[1 - f^\varepsilon(z+1, s)] \tag{5.2}$$

and the evolution kernel $G_N^{(x,t)}$ is the solution of the following equation:

$$\begin{aligned}
&\frac{\partial G_N^{(x,t)}}{\partial s}(y, s) = \frac{1}{2} \Delta_1 G_N^{(x,t)}(y, s) - aF^\varepsilon(x, t)\, \Delta_N G_N^{(x,t)}(y, s) \\
&G_N^{(x,t)}(y, 0) = \delta(y)
\end{aligned} \tag{5.3}$$

where $\delta(y)$ denotes the function which equals 1 at $n(2\varepsilon^{-2}+1)$, $n \in \mathbb{Z}$, and 0 otherwise. Then given any $\tau > 0$, there is a c such that for all x, all $t \leqslant \varepsilon^{-2}\tau$, and all $\varepsilon > 0$

$$|f^\varepsilon(x, t) - \Phi^\varepsilon(x, t)| \leqslant ct\varepsilon^{2+\alpha} \tag{5.4}$$

Proof of Lemma 5.1. We have

$$\frac{\partial f^\varepsilon(x,t)}{\partial t} - \left[\frac{1}{2}\Delta_1 f^\varepsilon(x,t) - aF^\varepsilon(x,t)\,\Delta_N f^\varepsilon(x,t) - a\Gamma(x,t)\,\nabla_1^- F^\varepsilon(x,t)\right] = E^\varepsilon(x,t)$$

where

$$\Gamma(x,t) = \frac{1}{N^2}\sum_{i=1}^{N} [f^\varepsilon(x-1+i,t) - f^\varepsilon(x-1-i,t)]$$

$$|E^\varepsilon(x,t)| \leqslant c\varepsilon^3 N$$

as follows from a Taylor expansion up to third order and from the fact that $\bar{f}^\varepsilon$ has third derivatives which are uniformly bounded (c is a suitable constant). By using the variation of the constants formula with respect to the semigroup $G_N^{(x,t)}$ we then obtain that

$$|f^\varepsilon(x,t) - \Phi^\varepsilon(x,t)| \leqslant \int_0^t ds \sum_z |G_N^{(x,t)}(z-x, t-s)\,E^\varepsilon(z,s)|$$

By using the above estimate on E^ε and the fact that for any given $\tau > 0$ there is a constant c such that

$$\sum_z |G_N^{(x,t)}(z-x, t-s)| \leqslant c$$

for all $x, s \leqslant t \leqslant \varepsilon^{-2}\tau$ and $\varepsilon > 0$. This statement will be proven in Section 6; therefore Lemma 5.1 is proven. ∎

Proof of Proposition 4.2. We rewrite (4.2) as

$$\frac{\partial\rho}{\partial t}(x,t) = \frac{1}{2}\Delta_1\rho(x,t) - a\nabla_1^-\{\rho(x,t)[1-\rho(x+1,t)]\,\Lambda(x,t)\}$$

where $\Lambda(x,t)$ is defined in (4.2c).

Notice that the maximum principle is not valid for Eq. (4.2), so that even though $\rho(x, nT\,|\,\underline{\eta}) \leqslant 1$ for all x and n, we cannot conclude that this holds also for $\rho(x,t\,|\,\underline{\eta})$ at all t. Let $t_0 > 0$ be such that for all $t \leqslant t_0$, $\rho(x,t\,|\,\underline{\eta}) \leqslant 2$. We shall prove in the sequel that for any fixed $T > 0$ we have that $|\rho(x,t\,|\,\underline{\eta}) - f^\varepsilon(x,t)| \leqslant c\varepsilon^\alpha$ for some $\alpha > 0$ and all $t \leqslant t_0$, $t \leqslant \varepsilon^{-2}T$. Since $f^\varepsilon \leqslant 1$ [because of the maximum principle, which applies to (1.1)] we have that up to the minimum between t_0 and $\varepsilon^{-2}T$, $\rho(x,t\,|\,\underline{\eta}) \leqslant 1 + c\varepsilon^\alpha$, so that for ε small, $t_0 \geqslant \varepsilon^{-2}T$. For this reason we can and shall assume hereafter that $\rho(x,t\,|\,\underline{\eta}) \leqslant 2$.

We add and subtract

$$aF^{\varepsilon}(x, t)\nabla_1^- A(x, t)$$

and we obtain

$$\frac{\partial\rho}{\partial t}(x, t) = \frac{1}{2}\Delta_1\rho(x, t) - aF^{\varepsilon}(x, t)\nabla_1^- A(x, t)$$
$$- a\nabla_1^-[\{\rho(x, t)[1-\rho(x+1, t)] - F^{\varepsilon}(x, t)\} A(x, t)]$$
$$- A(x-1, t)\nabla_1^- F^{\varepsilon}(x, t)$$

Taking into account that

$$\nabla_1^- A(x, t) = \Delta_N\rho(x, t) + \frac{1}{N}\nabla_N^-\nabla_1^-\rho(x)$$

we write an integral representation for ρ with respect to the same evolution kernel $G_N^{(x,t)}$ which appears in Lemma 5.1. Namely, given $\underline{\eta}$ as in Proposition 4.2, we have, for $kT < t \leqslant (k+1)T$,

$$\rho(x, t) = \sum_z G_N^{(x,t)}(z-x, t-kT)\,\eta(z, kT)$$
$$+ a\int_{kT}^{t} ds\, J\sum_z G_N^{(x,t)}(z-x, t-s)[F^{\varepsilon}(x, t) - F^{\varepsilon}(z, s)]\,\Delta_N\rho(z, s)$$
$$- a\int_{kT}^{t} ds\sum_z G_N^{(x,t)}(z-x, t-s)$$
$$\times\nabla_1^-[\{\rho(z, s)[1-\rho(z+1, s)] - F^{\varepsilon}(z, s)\} A(z, s)]$$
$$- a\int_{kT}^{t} ds\sum_z G_N^{(x,t)}(z-x, t-s)\,A(z-1, s)\,\nabla_1^- F^{\varepsilon}(z, s)$$
$$- a\int_{kT}^{t} ds\sum_z G_N^{(x,t)}(z-x, t-s)\frac{1}{N}[\nabla_1^-\nabla_N^-\rho(z, s)]\,F^{\varepsilon}(z, s) \qquad (5.5)$$

Here and in the following, as shorthand for $\rho(z, s\,|\,\underline{\eta})$ we use $\rho(z, s)$. We define

$$h^{\varepsilon}(x, t) = f^{\varepsilon}(x, t) - \rho(x, t) \qquad (5.6)$$

Then, taking into account Lemma 5.1, we have

$$\bigg| h^{\varepsilon}(x, t) - \sum_z G_N^{(x,t)}(z-x, t)[\eta(z, kT) - f^{\varepsilon}(z, kT)]$$
$$+ a\int_{kT+\varepsilon^{-1/2}}^{t} \sum_z G_N^{(x,t)}(z-x, t-s)\sum_{i=1}^{6} A_i(z, s)\bigg|$$
$$\leqslant c(\varepsilon^{1/2-\alpha} + \varepsilon^{\alpha+\beta}) \qquad (5.7)$$

where c is a constant (whose value changes from line to line)

$$A_1(z, s) = [F^\varepsilon(x, t) - F^\varepsilon(z, s)]\, \Delta_N h^\varepsilon(z, s) \tag{5.8a}$$

[A_1 comes from the difference between the first integral on the right-hand side of (5.1) and that on the right-hand side of (5.5)]

$$A_2(z, s) = \frac{1}{N^2} \sum_{i=1}^{N} [h^\varepsilon(z-1+i, s) - h^\varepsilon(z-1-i, s)]\, \nabla_1^- [F^\varepsilon(z, s)] \tag{5.8b}$$

[A_2 comes from the third integral in (5.5) and the second one in (5.1)]

$$A_3(z, s) = \frac{1}{N^2} \nabla_1^- \left\{ \sum_{i=1}^{N} [h^\varepsilon(z+i, s) - h^\varepsilon(z-i, s)]\, H^\varepsilon(z, s) \right\} \tag{5.8c}$$

$$H^\varepsilon(z, s) = h^\varepsilon(z, s)[1 + Jh^\varepsilon(z+1, s)] - f^\varepsilon(z, s)\, h^\varepsilon(z+1, s) - h^\varepsilon(z, s)\, f^\varepsilon(z+1, s)$$

$$A_4(z, s) = \frac{1}{N^2} \nabla_1^- \left\{ \sum_{i=1}^{N} [f^\varepsilon(z+i, s) - f^\varepsilon(z-i, s)]\, H^\varepsilon(z, s) \right\} \tag{5.8d}$$

[A_3 and A_4 come from the second integral in (5.5), after adding and subtracting f^ε to the corresponding ρ]

$$A_5(z, s) = \frac{1}{N} [\nabla_1^- \nabla_N^- h^\varepsilon(z, s)]\, F^\varepsilon(z, s) \tag{5.8e}$$

$$A_6(z, s) = \frac{1}{N} [\nabla_1^- \nabla_N^- f^\varepsilon(z, s)]\, F^\varepsilon(z, s) \tag{5.8f}$$

A_5 and A_6 take into account the contribution of the last integral in (5.5). Finally, the term $c\varepsilon^{1/2-\alpha}$ on the right-hand side of the inequality (5.7) arises from our having changed the interval of integration to $[kT + \varepsilon^{-1/2}, t]$: since the integrand can be bounded by c/N, we then get the estimate $c\varepsilon^{1/2-\alpha}$. The term $c\varepsilon^{\alpha+\beta}$ comes from (5.4).

Furthermore, we have that

$$\begin{aligned} &\sum_z G_N^{(x,t)}(z-x, t-kT)[\eta(z, kT) - f^\varepsilon(z, kT)] \\ &\quad = \sum_z G_N^{(x,t)}(z-x, t-kT)\, h(z, kT) \\ &\qquad + \sum_z G_N^{(x,t)}(z-x, t-kT)[\eta(z, kT) - \rho(z, kT; \eta_{(k-1)T})] \end{aligned} \tag{5.9}$$

where $\rho(z, t; \eta)$ is the solution of (4.2) at (z, t) when the initial datum is η. For the last term in (5.9) we get

$$\left|\sum_z G_N^{(x,t)}(z-x, t-kT)[\eta(z, kT)-\rho(z, kT; \eta_{(k-1)T})]\right| \leqslant c\varepsilon^{\zeta} \quad (5.10)$$

To prove (5.10), we write

$$G_N^{(x,t)}(y, s) = e^{Bs}e^{As}(y)$$

where

$$A = (\tfrac{1}{2} - K)\,\Delta_1$$
$$B = K\Delta_1 - aF^{\varepsilon}(x, t)\,\Delta_N$$
$$\tfrac{1}{2} > K > a \sup_{(x,t)} F^{\varepsilon}(x, t)$$

[the choice of K above is made possible by the assumptions on the initial datum and the validity of the maximum principle for (1.1)]. We shall prove in Proposition 6.1 that for all $s \leqslant \varepsilon^{-2}\tau$ and $\varepsilon > 0$

$$\sum_y |e^{Bs}(y)| \leqslant c$$

Using this, the fact that η is chosen as in Proposition 4.2, and that $t - kT \geqslant \varepsilon^{-1/2}$, we derive (5.10). By (5.10) we then have from (5.7)

$$\left|h^{\varepsilon}(x, t) - \sum_z G_N^{(x,t)}(z-x, t)\, h^{\varepsilon}(z, kT) + a\int_{kT+\varepsilon^{-1/2}}^{t} ds \sum_z G_N^{(x,t)}(z-x, t-s) \sum_{i=1}^{6} A_i(z, s)\right|$$
$$\leqslant c \max\{\varepsilon^{1/2-\alpha}, \varepsilon^{\alpha+\beta}, \varepsilon^{\zeta}\} \quad (5.11)$$

Iterating (5.11), we get

$$\left|h^{\varepsilon}(x, t) + a\sum_{i=1}^{6} \int_{I(t)} ds \sum_z G_N^{(x,t)}(z-x, t-s)\, A_i(z, s)\right|$$
$$\leqslant c \max\{\varepsilon^{1/2-\alpha-\beta}, \varepsilon^{\zeta-\beta}, \varepsilon^{\alpha}\} \quad (5.12)$$

where

$$I(t) = [0, t] \Big\backslash \bigcup_{k=0}^{n(t)} [kT, kT+\varepsilon^{-1/2}) \quad (5.13)$$

and $n(t)$ is the largest integer such that $n(t)T < t$. By choosing α and β positive but sufficiently small, we make the right-hand side of (5.12) infinitesimal, as $\varepsilon \to 0$. From (5.12) we get

$$|h^\varepsilon(x, t)| < a \sum_{i=1}^{6} \int_{I(t)} ds \left| \sum_z G_N^{(x,t)}(z-x, t-s)\, A_i(z, s) \right| + c \max\{\varepsilon^{1/2-\alpha-\beta}, \varepsilon^{\zeta-\beta}, \varepsilon^{\alpha}\} \tag{5.14}$$

As in the proof of Theorem 2.1, we shall estimate the integral in (5.14) by integrating by parts the derivatives which are present in the A_i terms. Doing this, we get derivatives acting on G_N, which we are able to estimate as in the continuous case; this will require some lengthy computations which are postponed to Sections 6 and 7. Setting

$$h_s = \sup_z |h(z, s)|$$

we have

$$\left| \sum_z G_N^{(x,t)}(z-x, t-s)\, A_1(z, s) \right| \leqslant c \left[\frac{\varepsilon}{(t-s)^{1/2}} + \varepsilon^2 \right] h_s \tag{5.15a}$$

which will be proven in Lemma 7.1, and

$$\begin{aligned} &\left| \sum_z G_N^{(x,t)}(z-x, t-s)\, A_2(z, s) \right| \\ &\quad \leqslant \frac{\varepsilon}{N^2} \left| \sum_z G_N^{(x,t)}(z-x, t-s) \sum_{i=1}^{N} [h^\varepsilon(z+i, s) - h^\varepsilon(z-i, s)] \right| \\ &\quad = \frac{\varepsilon}{N^2} h_s \sum_{i=1}^{N} \sum_z |G_N^{(x,t)}(x-i-z, t-s) - G_N^{(x,t)}(x+i-z, t-s)| \\ &\quad \leqslant c\varepsilon \frac{h_s}{(t-s)^{1/2}} \end{aligned} \tag{5.15b}$$

The last inequality comes from Proposition 6.4.

The next estimate holds for $kT + \varepsilon^{-1/2} < s \leqslant (k+1)T$:

$$\begin{aligned} &\left| \sum_z G_N^{(x,t)}(z-x, t-s)\, A_3(z, s) \right| \\ &\quad \leqslant c \frac{h_s}{(t-s)^{1/2}} \left[\frac{\varepsilon^\zeta + h_{kT}}{(s-kT-\varepsilon^{-1/2})^{1/2}} + \varepsilon^{1-\alpha-2\delta} h_s^2 + \varepsilon^{1-2\delta} h_s \right] \end{aligned} \tag{5.15c}$$

For a proof see Lemma 7.2 (proven under the assumption that $\alpha < 1/2$ and that δ is a suitably small positive constant). We also have

$$\left|\sum_z G_N^{(x,t)}(z-x, t-s)\, A_4(z,s)\right|$$
$$\leqslant c\varepsilon h_s \sum_z |\nabla_1^+ G_N^{(x,t)}(z-x, t-s)| \leqslant c\varepsilon \frac{h_s}{(t-s)^{1/2}} \tag{5.15d}$$

which is obtained from Proposition 6.3 and from the fact that

$$N^{-2} \sum_{i=1}^{N} |[f^\varepsilon(z+i,s) - f^\varepsilon(z-i,s)]| \leqslant c\varepsilon$$

Integrating by parts, we easily obtain that

$$\left|\sum_z G_N^{(x,t)}(z-x, t-s)\, A_5(z,s)\right|$$
$$\leqslant \frac{c}{N^2} h_s \left\{\sum_z |\nabla_1^+ G_N^{(x,t)}(z-x, t-s)| + \varepsilon \sum_z |G_N^{(x,t)}(z-x, t-s)|\right\}$$
$$\leqslant \frac{c}{N^2}\left[\frac{h_s}{(t-s)^{1/2}} + \varepsilon h_s\right] \tag{5.15e}$$

The last inequality follows from Propositions 6.1 and 6.4:

$$\left|\sum_z G_N^{(x,t)}(z-x, t-s)\, A_6(z,s)\right| \leqslant c\varepsilon^{3-\alpha} \tag{5.15f}$$

which easily follows from Proposition 6.1. We introduce the function

$$v(s) = kT, \qquad kT \leqslant s < (k+1)\, T$$

Then, collecting the estimates in (5.15), we have from (5.14)

$$h_t^\varepsilon \leqslant c \int_{I(t)} ds\, h_s \left[\frac{1}{(t-s)^{1/2}}\left(\varepsilon + \frac{\varepsilon^\zeta + h_{v(s)}}{[s - v(s) - \varepsilon^{-1/2}]^{1/2}} + \varepsilon^{1-\alpha-2\delta} h_s^2 + \varepsilon^{1-2\delta} h_s\right) + \varepsilon^2\right]$$
$$+ c \max\{\varepsilon^{1/2-\alpha-\beta}, \varepsilon^{\zeta-\beta}, \varepsilon^\alpha\} \tag{5.16}$$

We now fix $C \geqslant 1$ as in (5.20) below and we prove that it is a contradiction to assume that there is a first time $t_1 \leqslant \varepsilon^{-2}\tau$ for which $h^{\varepsilon}_{t_1} \geqslant C\varepsilon^{\lambda}$, where

$$\lambda = \min\{1/2 - \alpha - \beta, \zeta - \beta, \alpha\} \tag{5.17}$$

In the integral in (5.16) we bound the factors h^i_s by $h_s(C\varepsilon^{\lambda})^{i-1}$, $i \geqslant 1$, and we get

$$h^{\varepsilon}_{t_1} \leqslant c'\varepsilon \int_{I(t_1)} ds\, h_s \frac{1}{(t_1 - s)^{1/2}} (1 + 4\varepsilon^{\gamma} C^2) + c\varepsilon^{\lambda} \tag{5.18a}$$

$$\gamma = \min\{2\lambda - 2\delta - \alpha, \lambda - 2\delta\} \tag{5.18b}$$

$$c' \geqslant 1 \quad \text{is such that} \quad c' > c \quad \text{and} \quad c\varepsilon \leqslant \frac{c'}{(t_1 - s)^{1/2}} \tag{5.18c}$$

We iterate (5.18a) and obtain a series whose terms we estimate using (4.21c). We get after some straightforward computation

$$h^{\varepsilon}_{t_1} \leqslant c'\varepsilon^{\lambda} \exp\left[\frac{\pi}{2}\tau c'^2 (1 + 4C^2\varepsilon^{\gamma})^2\right] [1 + 2c'(1 + 4C^2\varepsilon^{\gamma})\,\tau^{1/2}] \tag{5.19}$$

For ε so small that

$$(1 + 4C^2\varepsilon^{\gamma}) \leqslant 2$$

(5.19) contradicts the assumption that $h^{\varepsilon}_{t_1} \geqslant C\varepsilon^{\lambda}$, because

$$C = 2c' \exp\left(\frac{\pi}{2}\tau c'^2 4\right)(1 + 2c' 2\tau^{1/2}) \tag{5.20}$$

6. UNIFORM l_1 ESTIMATES ON $G_N^{(x,t)}$

We consider the semigroup $\exp\{A_N t\}$, where

$$A_N = \Delta_1 - b\Delta_N$$
$$0 < b < 1 \tag{6.1}$$
$$N = \text{integer part of } \varepsilon^{-1+\alpha}, \quad 0 < \alpha < 1$$

acting on the set of all the periodic functions on the lattice which have period $|\Gamma^{\varepsilon}| = 2\varepsilon^{-2}$ (we only consider values of ε for which ε^{-2} is an integer). We are going to prove the following proposition.

Proposition 6.1. For any given $\tau > 0$ there is c so that for all $\varepsilon > 0$ and all $t \leqslant \varepsilon^{-2}\tau$

$$\sup_{y} \sum_{x=1}^{|\Gamma^\varepsilon|} |e^{A_N t}(y, x)| \leqslant c \tag{6.2}$$

where $\exp\{A_N t\}(y, x)$ denotes the kernel of the operator $\exp\{A_N t\}$.

We present here a proof due to P. M. Bleher (except for some minor modifications); we thank him for this and for many helpful discussions.

Recall that the kernel $\exp\{A_N t\}(y, x)$ is defined as

$$e^{A_N t}(y, x) = (\delta_x, e^{A_N t}\delta_y)$$

where $\delta_x(z)$ equals 1 if $z = x + n\,|\Gamma^\varepsilon|$, $n \in \mathbb{Z}$, and 0 otherwise. Notice also that $\exp\{A_N t\}$ is not a Markov semigroup, because $b > 0$; hence $\exp\{A_N t\}(y, x)$ is not necessarily positive. Finally, using translational invariance, we may and shall fix $y = 0$ in (6.2).

A last remark before the proof of the proposition, namely that we can apply it to the cases considered in the previous section because $G_N^{(x,t)}$ is as in (5.1). In fact, according to our assumptions, we can take $b = 2aF(x, t) < 1$ because the pair (x, t) is fixed.

The proof of Proposition 6.1 will be obtained by studying the spectrum of $\exp\{A_N t\}$. Recalling that we consider periodic functions with period $|\Gamma^\varepsilon| = 2\varepsilon^{-2}$, we call $\Gamma^\varepsilon = [1, 2\varepsilon^{-2}]$ and we introduce the Hilbert space

$$l^2(\Gamma^\varepsilon) = \left\{ f\colon \Gamma^\varepsilon \to \mathbb{R},\ \sum_{x \in \Gamma^\varepsilon} |f(x)|^2 < \infty \right\}$$

The eigenvectors of A_N are the harmonic functions on the lattice

$$\psi_k(x) = \frac{1}{|\Gamma^\varepsilon|^{1/2}} \exp\left\{ -i2\pi \frac{k}{|\Gamma^\varepsilon|} x \right\} \tag{6.3}$$

For $k \in \Gamma^\varepsilon$ they form an orthonormal basis providing the usual space-momentum representation:

$$f(x) = \sum_{k \in \Gamma^\varepsilon} \hat{f}(k)\, \psi_k(x) \tag{6.4a}$$

$$\hat{f}(k) = \sum_{x \in \Gamma^\varepsilon} \psi_k(x)\, f(x) \tag{6.4b}$$

The spectrum of A_N is therefore

$$\lambda_N(k) = 2\left[\left(\cos\frac{2\pi k}{|\Gamma^\varepsilon|} - 1\right) - \frac{b}{N^2}\left(\cos\frac{2\pi kN}{|\Gamma^\varepsilon|} - 1\right)\right]$$

$$= -4\left(\sin^2\frac{\pi k}{|\Gamma^\varepsilon|} - \frac{b}{N^2}\sin^2\frac{\pi kN}{|\Gamma^\varepsilon|}\right), \qquad k \in \Gamma^\varepsilon \tag{6.5}$$

To underline the dependence on b we shall sometimes write $\lambda_N(k; b)$. It is not difficult to see that, for all $k \in \Gamma^\varepsilon$, $\lambda_N(k) \leqslant \lambda_1(k) \leqslant 0$. We shall, however, need and prove a weaker statement, as follows.

Lemma 6.2. For any $0 < b < b' < 1$ and for ε small enough the following holds. For all $k \in \Gamma^\varepsilon$: $\lambda_N(k, b) \leqslant \lambda_1(k, b') \leqslant 0$.

Proof. Obviously $\lambda_1(k, b') \leqslant 0$ because $b' < 1$. By (6.5) the proof that $\lambda_N(k, b) \leqslant \lambda_1(k, b')$ is reduced to proving that for any $0 < b < 1$, $\lambda_N(k, b) \leqslant 0$. We start by considering the values of k for which

$$\frac{\pi kN}{|\Gamma^\varepsilon|} \leqslant \frac{\pi}{2}, \qquad \text{i.e.,} \quad \frac{\pi k}{|\Gamma^\varepsilon|} \leqslant \frac{\pi}{2N}$$

We call $\mathscr{I}_1$ such an interval. Given any $0 < d < 1$, there exists N so large that for $k \in \mathscr{I}_1$

$$\sin\frac{\pi k}{|\Gamma^\varepsilon|} \geqslant d\frac{\pi k}{|\Gamma^\varepsilon|}$$

Hence

$$\sin^2\frac{\pi k}{|\Gamma^\varepsilon|} - \frac{b}{N^2}\sin^2\frac{\pi kN}{|\Gamma^\varepsilon|} \geqslant \left(\frac{d\pi k}{|\Gamma^\varepsilon|}\right)^2 - \frac{b}{N^2}\sin^2\frac{\pi kN}{|\Gamma^\varepsilon|}$$

which is nonnegative if we choose $d \geqslant b$ ($\sin^2 x \leqslant x^2$). In the next interval, $\mathscr{I}_2$, of values of k

$$\frac{\pi}{2} \leqslant \frac{\pi kN}{|\Gamma^\varepsilon|} \leqslant \pi$$

the last term on the right-hand side of the last equality in (6.5) takes on the same values as in $\mathscr{I}_1$, while the other term on the same side of the equality has values larger than in $\mathscr{I}_1$. The same keeps happening until $k \leqslant |\Gamma^\varepsilon|/2$: therefore (6.5) is proven for such values of k, and by symmetry the proof extends then to all the values of k. ∎

By translational invariance we fix $y=0$ in $\exp\{A_N t\}(y, x)$, which we will write as $G_N(x; t)$. We have

$$G_N(x; t) = \frac{1}{|\Gamma^\varepsilon|^{1/2}} \sum_{k \in \Gamma^\varepsilon} e^{\lambda_N(k)t} \psi_k(x) \tag{6.6}$$

$\psi_k(x)$ being defined in (6.3). Its l^2 norm is

$$\|G_N(\cdot; t)\|_2 = \frac{1}{|\Gamma^\varepsilon|^{1/2}} \left(\sum_{k \in \Gamma^\varepsilon} e^{2\lambda_N(k)t} \right)^{1/2} \tag{6.7}$$

Given any $b' < 1$, there is c so that

$$\frac{\sqrt{t}}{|\Gamma^\varepsilon|} \sum_k \exp\{\lambda_1(k, b')t\} < c$$

for all $\varepsilon > 0$ and $t \leqslant \varepsilon^{-2}\tau$, because the above is the Riemann sum of an integrable function. Using Lemma 6.2, for ε small enough we get that

$$\|G_N(\cdot; t)\|_2 \leqslant \frac{c}{t^{1/4}}$$

Hence

$$\begin{aligned}\sum_x |G_N(x; t)| &= \sum_{x \leqslant \sqrt{t}} |G_N(x; t)| + \sum_{x > \sqrt{t}} |G_N(x; t)| \\ &\leqslant t^{1/4} \|G_N(\cdot; t)\|_2 + \sum_{x > \sqrt{t}} |G_N(t, x)|\end{aligned}$$

In order to bound the last sum, we mimic the proof for the continuous case, where one can derive the decay in x of $G_N(t, x)$ from the smoothness of its Fourier transform by the Riemann–Lebesgue theorem. With this in mind we write

$$\frac{\nabla_1^\pm e^{-iks}}{e^{\mp is} - 1} = e^{-iks}$$

where the derivative acts on the k variable. We have also set $s = 2\pi x/|\Gamma^\varepsilon|$. We integrate twice by parts the right-hand side of (6.6). Taking into account that

$$|(e^{\pm is} - 1)| \geqslant s, \qquad 0 \leqslant s \leqslant \pi$$

we have the following estimate for $G_N(x; t)$, $|x| \leqslant |\Gamma^\varepsilon|$:

$$|G_N(x; t)| \leqslant \frac{c}{x^2} |\Gamma^\varepsilon| \sum_{k \in \Gamma^\varepsilon} |\Delta_1 e^{\lambda_N(k)t}| \tag{6.8}$$

where, again, Δ_1 acts on k. We write

$$\Delta_1 e^{\lambda_N(k)t} = e^{\lambda_N(k)t}[e^{t\nabla_1^+ \lambda_N(k)} + e^{-t\nabla_1^- \lambda_N(k)} - 2]$$

Since

$$|t\nabla_1^\pm \lambda_N(k)| \leqslant c \frac{tk}{|\Gamma^\varepsilon|^2} \leqslant c\tau \tag{6.9}$$

we get, using Lemma 6.2,

$$|\Delta_1 e^{\lambda_N(k)t}| \leqslant c e^{\lambda_1(k,b')t} \left[t\Delta_1 \lambda_N(k) + \left(\frac{tk}{|\Gamma^\varepsilon|^2} \right)^2 \right] \tag{6.10}$$

We have

$$|t\Delta_1 \lambda_N(k)| \leqslant c \frac{t}{|\Gamma^\varepsilon|^2}$$

and, for $\pi k/|\Gamma^\varepsilon| \leqslant \pi/2$ (by symmetry we can restrict ourselves to this case)

$$\lambda_1(k, b')t \leqslant -ct \frac{k^2}{|\Gamma^\varepsilon|^2}$$

We then get from (6.8)

$$|G_N(x; t)| \leqslant \frac{c}{x^2} |\Gamma^\varepsilon| \sum_{k \in \Gamma^\varepsilon} e^{-ctk^2/|\Gamma^\varepsilon|^2} \left[\frac{t}{|\Gamma^\varepsilon|^2} + \left(\frac{tk}{|\Gamma^\varepsilon|^2} \right)^2 \right]$$

Call

$$\delta = \frac{\sqrt{t}}{|\Gamma^\varepsilon|}$$

Then

$$|G_N(x; t)| \leqslant \frac{c}{x^2} \delta \sqrt{t} \sum_{k \in \Gamma^\varepsilon} e^{-c(\delta k)^2} [1 + (\delta k)^2] \tag{6.11}$$

Since the sum over $|x| \geqslant \sqrt{t}$ of $1/x^2$ gives a factor $1/\sqrt{t}$, we get a uniform bound on the l_1 norm of G_N, so that the proposition is proved. ∎

We conclude this section by proving a corollary of Proposition 6.1 which establishes properties used in Section 5.

Proposition 6.3. There is a constant c so that the following holds (we are using same notation and assumptions as before in this section):

$$\sum_y |\nabla_1^{\pm} G_N(y; t)| \leqslant \frac{c}{(t+1)^{1/2}} \tag{6.12}$$

$$\sum_y |\Delta_N G_N(y; t)| \leqslant \frac{c}{t+1} \tag{6.13}$$

$$\sum_y |\Delta_N G_N(y; t)|\ |x-y| \leqslant \frac{c}{(t+1)^{1/2}} \tag{6.14}$$

Proof. We write

$$G_N(y, t) = \sum_z e^{Bt}(z)\, e^{Ht}(y-z)$$

where

$$H = h\Delta_1, \qquad h > 0$$

$$B = (\tfrac{1}{2} - h)\, \Delta_1 - aF^{\varepsilon}(x, t)\, \Delta_N$$

$$\tfrac{1}{2} - h > a \sup_{(x,t)} F^{\varepsilon}(x, t)$$

and $\exp\{Bt\}(z)$ denotes the kernel of the operator $\exp\{Bt\}$ between 0 and z; here we are using translational invariance; $\exp\{Ht\}(z)$ is defined analogously. We have

$$\sum_y |\nabla_1^{\pm} G_N(y; t)| \leqslant \sum_z |e^{Bt}(z)| \sum_y |\nabla_1^{\pm} e^{Ht}(y-z)| \leqslant \frac{c}{(t+1)^{1/2}}$$

The last inequality comes from the uniform l_1 bound on e^{Bt} proven in Proposition 6.1 and the following classical estimate on symmetric random walks:

$$\sum_y |e^{Ht}(y) - G_t(y)| \leqslant \frac{c}{\sqrt{t}} \tag{6.15a}$$

where

$$G_t(y) = \frac{1}{(2\pi h t)^{1/2}}\, e^{-y^2/2ht} \tag{6.15b}$$

Equations (6.13) and (6.14) come also from classical properties of random walks; we give below an outline of the proof of (6.13), just for the sake of completeness. By the previous argument we reduce ourselves to proving (6.13) with G_N replaced by $\exp(Ht)$. We have to compute

$$\sum_y N^{-2} |e^{Ht}(y+N) + e^{Ht}(y-N) - 2e^{Ht}(y)| \tag{6.16}$$

Calling

$$M_t(y) = e^{Ht}(y) - G_t(y)$$

and setting $s = t/2$, we have

$$e^{Ht}(y) = \sum_z [M_s(z) + G_s(z)][M_s(y-z) + G_s(y-z)]$$

We develop the product of the two square brackets; the terms with two M's when summed over y, by (6.15a), give the desired estimate, c/t. The terms with two G's when inserted in (6.16) give again the c/t bound, as can be easily checked. We are left with the terms having an M and a G factor. By a change of variables we can always write first the M and then the G: when inserted in (6.16) this gives

$$\sum_z M_s(z) \sum_y N^{-2} |G_s(y+N-z) + G_s(y-N-z) - 2G_s(y-z)|$$

which is also bounded by c/t.

Similarly we prove (6.14). ∎

7. TECHNICAL LEMMAS

Lemma 7.1. Let $G_N^{(x,t)}(y, s)$ be the solution of (5.3); then

$$\left| \sum_y G_N^{(x,t)}(y-x, t-s)\, A_1(y, s) \right| \leqslant [\varepsilon/(t-s)^{1/2} + \varepsilon^2]\, h_s \tag{7.1}$$

where $A_1(y, s)$ is defined in (5.8a) and $h_s = \sup_y |h(y, s)|$.

Proof. Integrating by parts, we have

$$\begin{aligned}
\sum_y & G_N^{(x,t)}(y-x, t-s)\, A_1(y, s) \\
&= \sum_y \Delta_N G_N^{(x,t)}(y-x, t-s)[F^\varepsilon(y, s) - F^\varepsilon(x, t)]\, h^\varepsilon(y, s) \quad (\equiv I_1) \\
&+ \sum_y G_N^{(x,t)}(y-x, t-s)\, \Delta_N [F^\varepsilon(y, s) - F^\varepsilon(x, t)]\, h^\varepsilon(y, s) \quad (\equiv I_2) \\
&+ \sum_y \nabla_N^- G_N^{(x,t)}(y-x, t-s)\, \nabla_N^+ [F^\varepsilon(y, s) - F^\varepsilon(x, t)]\, h^\varepsilon(y, s) \quad (\equiv I_3) \\
&+ \sum_y \nabla_N^+ G_N^{(x,t)}(y-x, t-s)\, \nabla_N^- [F^\varepsilon(y, s) - F^\varepsilon(x, t)]\, h^\varepsilon(y, s) \quad (\equiv I_4)
\end{aligned} \tag{7.2}$$

By the smoothness of F^ε, we have that

$$|I_1| \leqslant c \sum_y |\Delta_N G_N^{(x,t)}(y-x, t-s)| \, (\varepsilon \, |y-x| + \varepsilon^2 \, |t-s|) \, h_s$$
$$\leqslant c[\varepsilon h_s/(t-s)^{1/2} + \varepsilon^2 h_s] \tag{7.3a}$$

The last inequality comes from (6.13) and (6.14) stated in Proposition 6.4.

Furthermore, from Proposition 6.1 we have that

$$|I_2| \leqslant c\varepsilon^2 h_s \tag{7.3b}$$

and from (6.12) of Proposition 6.3

$$|I_3| \leqslant c\varepsilon h_s/(t-s)^{1/2} \tag{7.3c}$$

$$|I_4| \leqslant c\varepsilon h_s/(t-s)^{1/2} \tag{7.3d}$$

Collecting the estimates (7.3), we prove (7.1). ▮

Lemma 7.2. Let $G_N^{(x,t)}(y, s)$ be the solution of (5.3); then for $kT < s \leqslant (k+1)T$ and $0 < \alpha < 1/2$ we have

$$\left|\sum_y G_N^{(x,t)}(y-x, t-s) \, A_3(y, s)\right|$$
$$\leqslant c \frac{h_s}{(t-s)^{1/2}} \left\{ \frac{\varepsilon^\zeta + h_{kT}}{(s-kT-\varepsilon^{-1/2})^{1/2}} + \varepsilon^{1-2\delta} h_s + \varepsilon^{1-\alpha-2\delta} h_s^2 \right\} \tag{7.4}$$

where $A_3(y, s)$ is defined in (5.8c) and $h_s = \sup_y |h(y, s)|$.

Proof. Integrating by parts, we have that

$$\left|\sum_y G_N^{(x,t)}(y-x, t-s) \, A_3(y, s)\right| \tag{7.5}$$
$$= \left|\sum_y \nabla_1^+ G_N^{(x,t)}(y-x, t-s) \frac{1}{N^2} \sum_{i=1}^N \, [h^\varepsilon(y+i, s) - h^\varepsilon(y-i, s)] \, H^\varepsilon(y, s)\right|$$

We write

$$h^\varepsilon(y+i, s) - h^\varepsilon(y-i, s) = \sum_{k=1}^{2i} \nabla_1^+ h^\varepsilon(y+i-k, s)$$

Taking in account that $\sup_y |H^\varepsilon(y, s)| \leqslant ch_s$ by Proposition 6.3, we bound (7.5) by

$$c \frac{1}{N^2} \sum_{i=1}^N \sum_{k=1}^{2i} \sum_y h_s \, |\nabla_1^+ G_N^{(x,t)}(y-x, t-s)| \, |\nabla_1^+ h(y+i-k, s)|$$
$$\leqslant c \frac{h_s}{(t-s)^{1/2}} \sup_y |\nabla_1^+ h(y, s)|$$

Using the estimate (7.6), stated in Proposition 7.3 below, we have (7.4). ▮

Lemma 7.3. For $kT<t\leqslant(k+1)T$ and $0<\alpha<1/2$ we have

$$\sup_y |\nabla_1^+ h(y,t)| \leqslant c\left\{\frac{\varepsilon^\zeta + h_{kT}}{(t-kT-\varepsilon^{-1/2})^{1/2}} + \varepsilon^{1-2\delta} h_t + \varepsilon^{1-\alpha-2\delta} h_t^2\right\} \tag{7.6}$$

Proof. Proceeding as in Lemma 5.1 and using the representation for ρ in (5.5), we have, for $kT<t\leqslant(k+1)T$,

$$\begin{aligned}|\nabla_1^+ h^\varepsilon(x,t)| \leqslant{}& \left|\sum_z \nabla_1^+ G_N^{(x,t)}(z-x, t-kT)[\eta(z,kT) - f^\varepsilon(z,kT)]\right| \\ &+ a\int_{kT}^t \sum_z \left|\nabla_1^+ G_N^{(x,t)}(z-x,t-s) \sum_{i=1}^6 A_i(z,s)\right| + ct\varepsilon^{\alpha+1+2}\end{aligned} \tag{7.7}$$

But

$$\begin{aligned}&\sum_z \nabla_1^+ G_N^{(x,t)}(z-x,t-kT)[\eta(z,kT)-f^\varepsilon(z,kT)] \\ &\quad=\sum_z \nabla_1^+ G_N^{(x,t)}(z-x,t-kT)\, h(z,kT) \\ &\qquad+\sum_z \nabla_1^+ G_N^{(x,t)}(z-x,t-kT)[\eta(z,kT)-\rho(z,kT\,|\,\eta_{(k-1)T})]\end{aligned}$$

Therefore (7.7) is bounded by

$$\begin{aligned}|\nabla_1^+ h^\varepsilon(x,t)| \leqslant{}& \Bigg|\sum_z \nabla_1^+ G_N^{(x,t)}(z-x,t-kT) \\ &\times\{h(z,kT) + [\eta(z,kT) - \rho(z,kT\,|\,\eta_{(k-1)T})]\}\Bigg| \\ &+ a\int_{kT}^t \left|\sum_z \nabla_1^+ G_N^{(x,t)}(z-x,t-s)\sum_{i=1}^6 A_i(z,s)\right| + ct\varepsilon^{\alpha+1+2}\end{aligned} \tag{7.8}$$

Taking into account that

$$\sum_z \nabla_1^+ G_N^{(x,t)}(z,s) = \sum_y \nabla_1^+ G_N^{(x,t)}(y,s/2)\sum_z G_N^{(x,t)}(y-z,s/2) \tag{7.9}$$

we gain from $\sum_y \nabla_1^+ G_N^{(x,t)}(y,s/2)$ a factor of order $1/\sqrt{s}$, while the other term can be estimated as before. More precisely, we estimate the first term in (7.8) using Proposition 6.3 and the ε^ζ continuity of ρ (see Proposition 4.1); the second term using (5.15) for A_i, $i \neq 3$ (for A_3; see below). Therefore the first term in the right-hand side of (7.8) is less than or equal to

$$\frac{1}{(t-kT-\varepsilon^{-1/2})^{1/2}} [h_{kT} + \|\eta_{kT} - \rho_{kT}\|]$$
$$\leqslant \frac{1}{(t-kT-\varepsilon^{-1/2})^{1/2}} [h_{kT} + \varepsilon^{\zeta}] \tag{7.10}$$

[The $\|\cdot\|$ is defined in (4.3).]

For A_3 we have, integrating by parts,

$$\left|\sum_z \nabla_1^+ G_N^{(x,t)}(z-x, t-s)\, A_3(z,s)\right|$$
$$= \left|\sum_z \Delta G_N^{(x,t)}(z-x, t-s) \frac{1}{N^2} \sum_{i=1}^{N} [h^\varepsilon(y+i, s) - h^\varepsilon(y-i, s)]\, H^\varepsilon(y,s)\right|$$
$$\leqslant c\left[\frac{h_s^2}{N(t-s)^{1-\delta}}\right] \tag{7.11}$$

where $\delta > 0$ is arbitrarily chosen. The last inequality comes from Proposition 6.3 and from $\sup_y |H^\varepsilon(y,s)| \leqslant ch_s$.

Taking into account that $N = \varepsilon^{-1+\alpha}$, we get (7.6). ∎

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